

Recognition of Unit Segment and Polyline Graphs is $\exists\mathbb{R}$ -Complete

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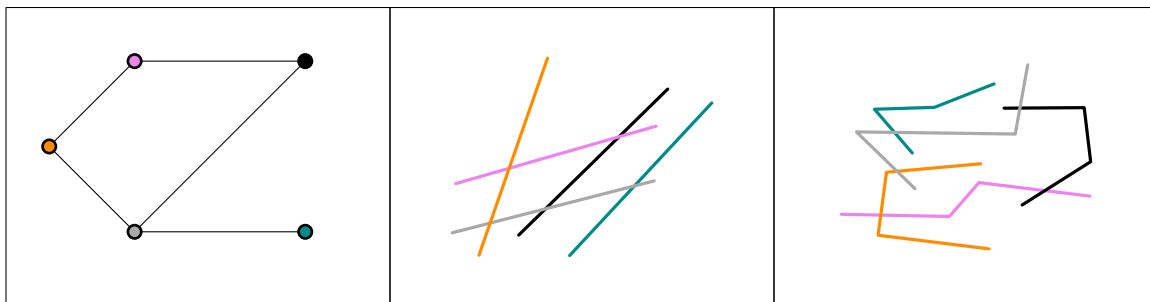
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Abstract

Given a set of objects O in the plane, the corresponding intersection graph is defined as follows. A vertex is created for each object and an edge joins two vertices whenever the corresponding objects intersect. We study here the case of unit segments and polylines with exactly k bends. In the recognition problem, we are given a graph and want to decide whether the graph can be represented as the intersection graph of certain geometric objects. In previous work it was shown that various recognition problems are $\exists\mathbb{R}$ -complete, leaving unit segments and polylines as few remaining natural cases. We show that recognition for both families of objects is $\exists\mathbb{R}$ -complete.



A graph and two representations as an intersection graph, of unit segments and polylines respectively.

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1 Introduction

Many real-life problems can be mathematically described in the language of graphs. For instance, consider all the cell towers in Switzerland. We want to assign each tower a frequency such that no two towers that overlap in coverage use the same frequency. This can be seen as a graph coloring problem: Every cell tower becomes a vertex, overlap indicates an edge and a frequency assignment corresponds to a proper coloring of the vertices, see Figure 1.

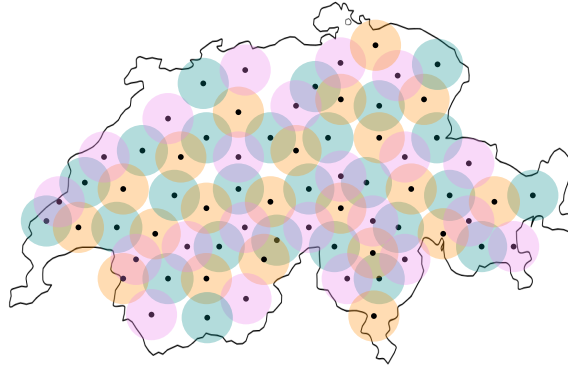


Figure 1: A fictional illustration of mobile coverage of Switzerland using cell towers.

In many contexts, we have additional structure on the graph that may or may not help us to solve the underlying algorithmic problem. For instance, it might be that the graph arises as the intersection graph of unit disks in the plane such as in the example above. In that case, the coloring problem can be solved more efficiently [10], and there are better approximation algorithms for the clique problem [11]. This motivates a systematic study of geometric intersection graphs. Another motivation is mathematical curiosity. Simple geometric shapes are easily visualized and arguably very natural mathematical objects. Studying the structural properties of intersection graphs gives insight into those geometric shapes and their possible intersection patterns.

It is known for a host of geometric shapes that it is $\exists\mathbb{R}$ -complete to recognize their intersection graphs [14, 25, 27, 28]. The class $\exists\mathbb{R}$ consists of all of those problems that are polynomial-time equivalent to deciding whether a given system of polynomial inequalities and equations with integer coefficients has a real solution. We will introduce $\exists\mathbb{R}$ in more detail below.

In this work, we focus on two geometric objects; unit segments and polylines with exactly k bends. Although we consider both types of geometric objects natural and well studied, to the best of our knowledge the complexity of their recognition problem was left open. We close this gap by showing that both recognition problems are $\exists\mathbb{R}$ -complete.

1.1 Definition and Results

We define geometric intersection graphs and the corresponding recognition problem.

Intersection graphs. Given a finite set of geometric objects O , we denote by $G(O) = (V, E)$, the corresponding *intersection graph*. The set of vertices is the set of objects ($V = O$) and two objects are adjacent ($uv \in E$) if they intersect ($u \cap v \neq \emptyset$). We are interested in intersection graphs that come from different families of geometric objects.

Families of geometric objects. Examples for a family of geometric objects are segments, disks, unit disks, unit segments, rays, and convex sets, to name a few of the most common ones. In general, given a geometric body $O \subset \mathbb{R}^2$ we denote by O^+ the family of all translates of O . Similarly, we denote by O^{\oplus} the

family of all translates and rotations of O . For example, the family of all unit segments can be denoted as u^{\oplus} , where u is a unit segment.

Graph classes. Classes of geometric objects \mathcal{O} naturally give rise to classes of graphs $C(\mathcal{O})$: Given a family of geometric objects \mathcal{O} , we denote by $C(\mathcal{O})$ the class of graphs that can be formed by taking the intersection graph of a finite subset from \mathcal{O} .

Recognition. If we are given a graph, we can ask if this graph belongs to a geometric graph class. Formally, let C be a fixed graph class, then the recognition problem for C is defined as follows. As input, we receive a graph G and we have to decide whether $G \in C$. We denote the corresponding algorithmic problem by $\text{RECOGNITION}(C)$. For example the problem of recognizing unit segment graphs can be denoted by $\text{RECOGNITION}(C(u^{\oplus}))$. We will use the term UNIT RECOGNITION for this problem. Furthermore, we define $k\text{-POLYLINE RECOGNITION}$ as the recognition problem of intersection graphs of polylines with k bends.

Realizations. We can also say that $\text{RECOGNITION}(C(\mathcal{O}))$ asks about the existence of a *representation* of a given graph. A representation or *realization* of a graph G using a family of objects \mathcal{O} is a function $r : V \mapsto \mathcal{O}$ such that $r(v) \cap r(w) \neq \emptyset \iff vw \in E$. For simplicity, for a set $V' \subseteq V$, we define $r(V') = \bigcup_{v \in V'} r(v)$.

Results. We show $\exists\mathbb{R}$ -completeness of the recognition problems of two very natural geometric graph classes.

Theorem 1. *UNIT RECOGNITION is $\exists\mathbb{R}$ -complete.*

Theorem 2. *$k\text{-POLYLINE RECOGNITION}$ is $\exists\mathbb{R}$ -complete, for any fixed $k \geq 1$.*

1.2 Discussion

In this section, we discuss strengths and limitations of our results from different perspectives. To supply the appropriate context, we give a comprehensive list of important geometric graph classes and the current knowledge about the complexity of their recognition problems in Table 1.

Table 1: Classes of Geometric Intersection Graphs and their algorithmic complexity.

| Intersection graphs of | Complexity | Source |
|----------------------------|-------------------------------|--|
| circle chords | polynomial | Spinrad [43] |
| (unit) interval | polynomial | Booth and Lueker [12] |
| string | NP-complete | Schaefer and Sedgwick [23, 36] |
| outerstring | NP-complete | Kratochvíl [22] (see also Rok and Walczak [32]) |
| C^+ , C convex polygon | NP-complete | Müller et al. [30], Kratochvíl, Matoušek [24] |
| (unit) disks | $\exists\mathbb{R}$ -complete | McDiarmid and Müller [28] |
| convex sets | $\exists\mathbb{R}$ -complete | Schaefer [33] |
| (downwards) rays | $\exists\mathbb{R}$ -complete | Cardinal et al. [14] |
| outer segments | $\exists\mathbb{R}$ -complete | Cardinal et al. [14] |
| segments | $\exists\mathbb{R}$ -complete | Kratochvíl and Matoušek [25] (see also [27, 33]) |
| unit balls | $\exists\mathbb{R}$ -complete | Kang and Müller [21] |

Refining the hierarchy. We see our main contribution in refining the hierarchy of geometric graph classes for which recognition complexity is known. Both unit segments as well as polylines with k bends are natural objects that are well studied in the literature. However, the recognition of the corresponding graph classes was not studied previously. Polylines with an unbounded number of bends are equivalent to strings (it is possible to show that polylines with an unbounded number of bends are as versatile as strings with respect

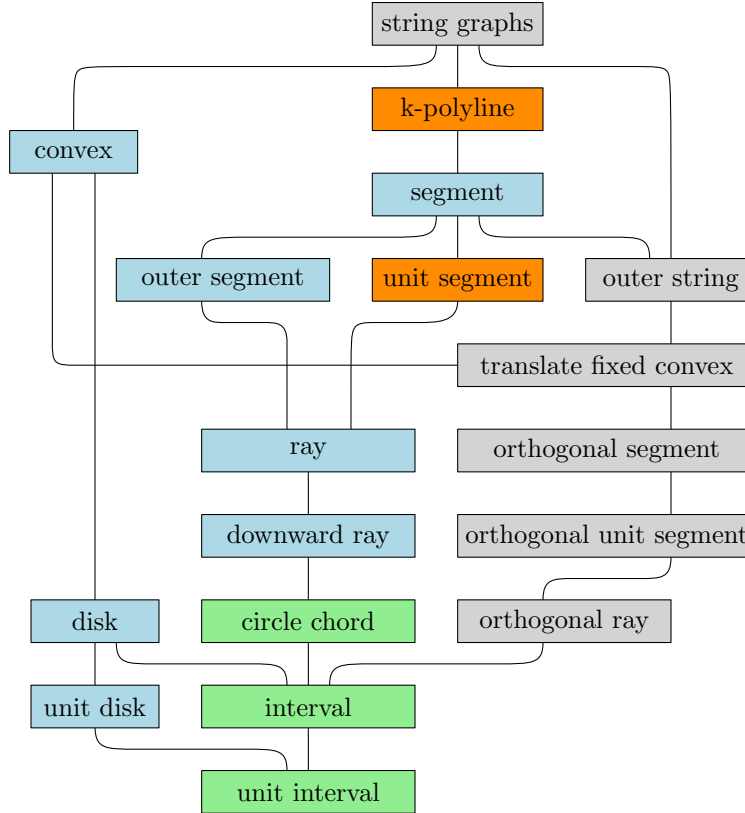


Figure 2: Each box represents a different geometric intersection graph class. Those marked in green can be recognized in polynomial time. Those in blue are known to be $\exists\mathbb{R}$ -complete. The ones in gray are NP-complete, and the orange ones are the new results presented in this paper.

to the types of graphs that they can represent, since the number of intersections of any two strings can always be bounded from above [36, 37]), while polylines with 0 bends are just segments. Polylines with k bends thus naturally slot in between strings and segments, and their corresponding graph class is thus also an intermediate class between the class of segment intersection graphs and string graphs, as can be seen in Figure 2. By showing that recognition for polylines with k bends is $\exists\mathbb{R}$ -complete for all constant k , we see that the switch from $\exists\mathbb{R}$ -completeness (segment intersection graphs) to NP-membership (string graphs) really only happens once k is infinite. Similarly, unit segment intersection graphs slot in between segment and ray intersection graphs. Intuitively, recognition of a class intermediate to two classes that are $\exists\mathbb{R}$ -hard to recognize should also be $\exists\mathbb{R}$ -hard, and our Theorem 1 confirms this intuition in this case.

Large coordinates. One of the consequences of $\exists\mathbb{R}$ -completeness is that there are no short representations of solutions known. Some representable graphs may only be representable by objects with irrational coordinates, or by rational coordinates with nominator and denominator of size at least $2^{2^{n^c}}$, for some fixed $c > 0$. In other words, the numbers to describe the position might need to be doubly exponentially large [27] for some graphs. For “flexible” objects like polylines, rational solutions can always be obtained by slightly perturbing the representation. For more “sturdy” objects like unit segments this may not be possible, however it is known that for example unit disks admit rational solutions despite their inflexibility [28].

Unraveling the broader story. Given the picture of Figure 2, we wish to start unraveling a deeper story. Namely, we aim to get a better understanding of when geometric recognition problems are $\exists\mathbb{R}$ -complete and

when they are contained in NP. Figure 2 indicates that $\exists\mathbb{R}$ -hardness comes from objects that are complicated enough to avoid a complete combinatorial characterization. Unit interval graphs, interval graphs and circle chord graphs admit such a characterization that in turn can be used to develop a polynomial-time algorithm for their recognition. On the other hand, if the geometric objects are too broad, the recognition problem is in NP. The prime example is string graphs. For string graphs, it is sufficient to know the planar graph given by the intersection pattern of the strings, which is purely combinatorial information that does not care anymore about precise coordinates. Despite the fact that there are graphs that need an exponential number of intersections, it is possible to find a polynomial-size witness [36] and thus we do not have $\exists\mathbb{R}$ -hardness. We want to summarize this as: recognition problems are $\exists\mathbb{R}$ -complete if the underlying family of geometric objects is at a sweet spot of neither being too simplistic nor too flexible.

When we consider Table 1 we observe two different types of $\exists\mathbb{R}$ -complete families. The first type of family encapsulates all rotations O^{\oplus} of a given object O (i.e., segments, rays, unit segments etc.). The second type of family contains translates and possibly homothets of geometric objects that have some curvature themselves (i.e., disks and unit disks). However in case we fix a specific object without curvature, i.e., a polygon, and consider all translations of it then the recognition problem also lies in NP [30]. Therefore, broadly speaking, curvature or rotation seem to be properties needed for $\exists\mathbb{R}$ -completeness and the lack of it seems to imply NP-membership. We wish to capture one part of this intuition in the following conjecture:

Conjecture A. *Let O be a convex body in the plane with at least two distinct points. Then $\text{RECOGNITION}(O^{\oplus})$ is $\exists\mathbb{R}$ -complete.*

We wonder if our intuition on curvature of geometric objects could be generalized as well. It seems plausible that $\text{RECOGNITION}(O^{\oplus})$ is $\exists\mathbb{R}$ -complete if and only if O has curvature.

Restricted geometric graph classes. A classic way to make an algorithmic problem easier is to consider it only on restricted input. In the context of $\exists\mathbb{R}$ and geometric graph recognition, we would like to mention the work by Schaefer [35], where it is shown that the recognition problem of segments is already $\exists\mathbb{R}$ -complete for graphs of bounded degree. We conjecture that the same is true for unit segments and polylines. However, this does not follow from our reductions since we create graphs of unbounded degree.

Conjecture B. *UNIT RECOGNITION is $\exists\mathbb{R}$ -complete already for graphs of bounded degree.*

Once this would be established it would be interesting to find the exact degree threshold when the problem switches from being difficult to being easy.

1.3 Existential Theory of the Reals.

The class of the existential theory of the reals $\exists\mathbb{R}$ (pronounced as ‘ER’) is a complexity class defined through its canonical problem ETR, which also stands for Existential Theory of the Reals. In this problem we are given a sentence of the form $\exists x_1, \dots, x_n \in \mathbb{R} : \Phi(x_1, \dots, x_n)$, where Φ is a well-formed quantifier-free formula consisting of the symbols $\{0, 1, x_1, \dots, x_n, +, \cdot, \geq, >, \wedge, \vee, \neg\}$, and the goal is to check whether this sentence is true.

The class $\exists\mathbb{R}$ is the family of all problems that admit a polynomial-time many-one reduction to ETR. It is known that $\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}$ [13]. The reason that $\exists\mathbb{R}$ is an important complexity class is that a number of common problems, mainly in computational geometry, have been shown to be complete for this class. Schaefer established the current name and pointed out first that several known NP-hardness reductions actually imply $\exists\mathbb{R}$ -completeness [33]. Early examples are related to recognition of geometric structures: points in the plane [29, 42], geometric linkages [1, 34], segment graphs [25, 27], unit disk graphs [21, 28], ray intersection graphs [14], and point visibility graphs [14]. In general, the complexity class is more established in the graph drawing community [16, 18, 26, 35]. Yet, it is also relevant for studying polytopes [17, 31], Nash-Equilibria [6, 8, 9, 20, 38], and matrix factorization problems [15, 39, 40, 41]. Other $\exists\mathbb{R}$ -complete problems are the Art Gallery Problem [3, 44], covering polygons with convex polygons [2], geometric packing [5] and training neural networks [4, 7].

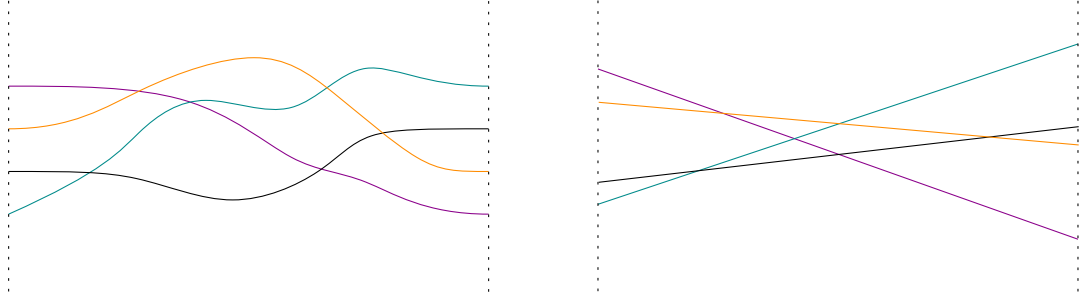


Figure 3: The pseudoline arrangement on the left is combinatorially equivalent to the (truncated) line arrangement on the right; hence, it is stretchable.

1.4 Proof Techniques

The techniques used in this paper are similar to previous work. $\exists\mathbb{R}$ -membership can be established straightforwardly by constructing concrete formulae or invoking a characterization of $\exists\mathbb{R}$ using *real* verification algorithms [19], similar to the characterization of NP. This proof can be found in Section 3.

For $\exists\mathbb{R}$ -hardness, we are in essence reducing from the STRETCHABILITY problem. In this problem, we are given a simple pseudoline arrangement as an input, and the question is whether this arrangement is stretchable. A pseudoline arrangement \mathcal{A} is a set of n curves that are x -monotone. Furthermore, any two curves intersect exactly once and no three curves meet in a single point. We assume that there exist two vertical lines on which each curve starts and ends. The problem is to determine whether there exists a combinatorially equivalent (truncated) line arrangement. See Figure 3 for an example.

For the reduction establishing $\exists\mathbb{R}$ -hardness of UNIT RECOGNITION, we are given a pseudoline arrangement \mathcal{A} , and we construct a graph that is representable by unit segments (or polylines with k bends) iff \mathcal{A} is stretchable. This graph is created by enhancing \mathcal{A} with more curves and taking their intersection graph. By the way these additional curves are constructed, they can always be drawn as unit segments in any line arrangement combinatorially equivalent to \mathcal{A} , if this line arrangement is first “squeezed” into a canonical form.

For the other direction of the equivalence we need to show that if the graph is representable by unit segments, then \mathcal{A} is stretchable. To help with this proof we establish helper lemmas for both unit segments and polylines stating that cycles in a graph enforce a certain order of intersections of objects around a closed curve in any realization of that graph. The details of these helper lemmas can be found in Section 2.

Using these additional lemmas we then show that the unit segments corresponding to the vertices originally representing our pseudolines must have the same combinatorial structure as our original arrangement \mathcal{A} . To ensure this we add so-called *probes* during the construction of our graph G . The ideas of order-enforcing cycles and probes have already been used in different contexts [14].

The main idea to show hardness of k -POLYLINE RECOGNITION is to enhance the construction for UNIT RECOGNITION by an additional area where the polylines representing our pseudolines have to make k bends. This then ensures that the polylines have no bends in the crucial part representing the pseudoline arrangement \mathcal{A} , and the same arguments as for unit segments can be used to show that this witnesses the stretchability of the original pseudoline arrangement.

2 Cycle Representations

In our reduction, we will construct a graph G that contains a cycle. The cycle helps us to enforce a certain structure on any geometric representation of G . The arguments in this section merely use that our geometric objects are Jordan arcs whose pairwise intersections consist of a finite number of connected components, and are not specific to either unit segments or polylines.

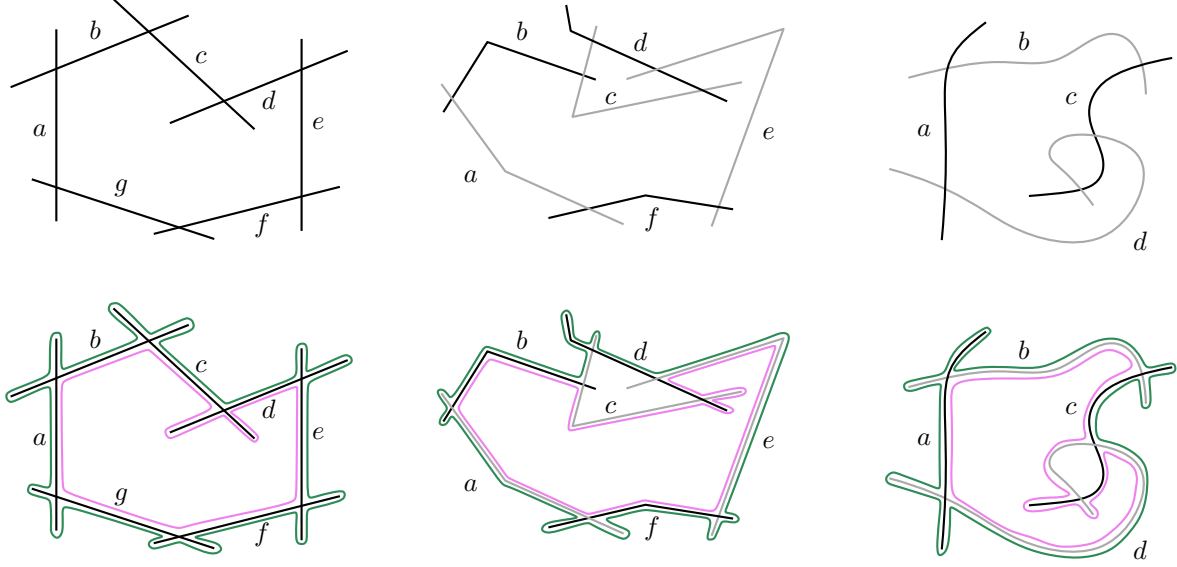


Figure 4: Top row: The representation $r(C)$ by segments, polylines and strings respectively. Bottom row: The representation of the top row with two boundary curves added. One on the interior and one on the exterior of the representation. From left to right we see unit segments, polylines with one bend and general Jordan arcs. Left: The interior curve traces the Jordan arcs $abcdcdefg$. The exterior curve traces $ababcbcdedefefgfgag$. Middle: The interior curve traces the Jordan arcs $abcdcdef$. The exterior curve traces $ababcbcddefefaf$. Right: The interior curve traces the Jordan arcs $abcdcdcd$. The exterior curve traces $ababcbcdad$.

We first want to introduce some notions about realizations of induced cycles by Jordan arcs. Let G be a graph, and let $C \subseteq V(G)$ be a set of vertices such that the induced graph $G[C]$ is a cycle (c_1, \dots, c_n) of $n \geq 4$ vertices. If we are now given a geometric representation of G as an intersection graph of Jordan arcs, we can define two Jordan curves “tracing” the representation $r(C)$ sufficiently close, one on the inside and one on the outside of $r(C)$. To be precise, by sufficiently close we mean that the curves are close enough to $r(C)$, such that between the two curves no other object $o \notin r(C)$ starts or ends, no proper crossing of two such other objects occurs, and such that every crossing of another object o with one of the curves implies that o also intersects $r(C)$ within a small ϵ -ball centered at the intersection with the curve. We call the two curves *interior boundary curve* and *exterior boundary curve*. See Figure 4 for an illustration.

Observation 3. *The interior boundary curve lies completely within the bounded cell bounded by the exterior boundary curve.*

Given either the interior or exterior boundary curve, we can record the *order of traced elements*, i.e., the order of elements $c \in C$ for which the boundary curve is close to $r(c)$, see Figure 4 for an illustration. This order admits some simple structure, that we describe in the following lemma.

Lemma 4. *Let C be an induced cycle of length $n \geq 4$, and let b be either the interior or exterior boundary curve of $r(C)$. Let $s \in C^*$ be the string describing the cyclic order of traced elements of b . Then this string can be decomposed into n consecutive parts s_1, \dots, s_n , where in each part s_i , only the elements c_i, c_{i+1} occur (note that we take indices modulo n).*

Proof. We stretch out the boundary curve horizontally and draw the Jordan arcs it traces on the top of the boundary curve (see Figure 5). Let us now assume that c_i appears again on the sequence after c_{i+2} was already seen, but not c_{i+3} yet. Note that the vertices c_{i+2} and c_i are not adjacent in C , thus their curves are not allowed to intersect. By definition the boundary curve is also not intersected. Thus c_{i+2} is enclosed

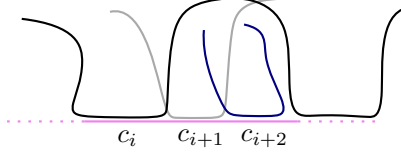


Figure 5: The Jordan arc c_{i+2} is encapsulated by the boundary curve and c_i and thus cannot intersect c_{i+3} .

by c_i and the boundary curve and cannot intersect c_{i+3} . This is a contradiction, and we thus know that for all i , c_i can only be seen before c_{i+2} or after c_{i+3} (with all indices taken modulo n).

We can thus split the string into n consecutive parts s_1, \dots, s_n by simply ending part s_i whenever c_i occurs for the last time. \square

Next, we want to argue that for our graph G the realizations of certain vertices of the graph must be contained in some cell.

2.1 Cell Lemma

We start by introducing the notion of a connector. Intuitively, it is just a bunch of vertices that separates an induced cycle from the (connected) rest of the graph.

Definition 5. Let $G = (V, E)$ be a graph, and $C \subseteq V$ be a set of vertices forming an induced cycle, such that $G[V \setminus C]$ is connected. Let $D \subseteq V \setminus C$ be a set of vertices with the following properties.

- The neighborhood of C is D .
- D is an independent set.
- $G[V \setminus D]$ consists of the two connected components, $G[C]$ and $G[V \setminus (C \cup D)]$.
- Each $d \in D$ has exactly one neighbor in C , and for any two distinct $d, d' \in D$, these neighbors are distinct and non-adjacent.

Then we call the set D *connectors of C* .

This definition is illustrated in Figure 6.

Lemma 6. Let C be an induced cycle of $G = (V, E)$ with connectors D and $r(V)$ be a geometric representation by Jordan arcs, then either $r(V \setminus (C \cup D))$ lies completely inside the interior curve of $r(C)$ or completely outside the exterior curve of $r(C)$.

Proof. This follows from the fact that $G[V \setminus (C \cup D)]$ is connected and is not adjacent to the induced cycle C . Thus it is also not intersecting the interior or exterior boundary curve. By the Jordan Curve Theorem, the interior (exterior) boundary curve splits the plane in two and thus $G[V \setminus (C \cup D)]$ can only be in one of the two components. \square

2.2 Order Lemma

The aim of this section is to show that certain vertices neighboring an induced cycle also need to intersect this cycle in the same order in any geometric representation. We start with a definition of the setup.

Definition 7. Let $G = (V, E)$ be a graph, and $C \subset V$ a set of vertices forming an induced cycle. Let $D \subset V \setminus C$ be a set of vertices with the following properties.

- Each $d \in D$ has either one or two neighbors in C .
- If $d \in D$ has two neighbors in C , these are non-adjacent.

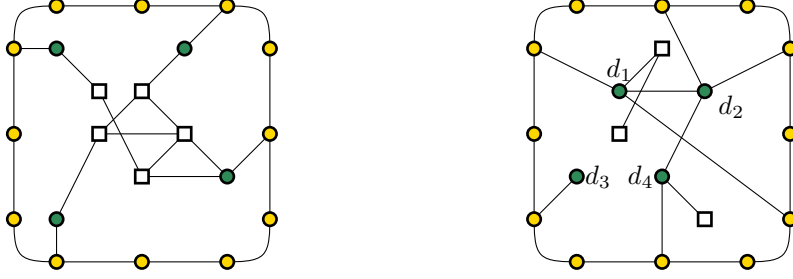


Figure 6: Left: The yellow dots form an induced cycle. The green vertices are connectors of this cycle and the remaining white squares form the rest of the graph. Right: The yellow dots form an induced cycle. The green dots are intersectors of this cycle. The cyclic order o_G as used in Lemma 10 is $d_2, d_2, d_1, d_4, d_3, d_1$, when starting at the top left of the cycle and going clockwise.

- For any two distinct $d, d' \in D$, their neighbors in C are distinct and non-adjacent.

Then we call the set D *intersectors of C* .

This definition is illustrated in Figure 6. We now define two cyclic orders of intersections of intersectors with the cycle. The first order lives in the realm of the graph, while the second one is concerned with a concrete realization. The goal of this section is to prove that these orders are the same.

Definition 8 (Graph Order of Intersectors). Let G be a graph with C an induced cycle on at least four vertices, and let D be intersectors of C . When we travel along the cycle C for one full rotation, we can write down the pairs $(c, d) \in C \times D$, such that $\{c, d\} \in E(G)$. This string of pairs defines the *graph order of the intersectors* up to a cyclic shift and reflection.

Definition 9 (Geometric Order of Intersectors). Let G be a graph with C an induced cycle on at least four vertices, and let D be intersectors of C . Let r be a realization of G by Jordan arcs and let b be the interior boundary curve of $r(C)$. When we travel along b for one full rotation, we can write down the pairs $(c, d) \in C \times D$ such that $r(d)$ intersects b where b is tracing $r(c)$. From consecutive copies of the same pair we only keep one. This string of pairs defines the *geometric order of the intersectors* up to a cyclic shift and reflection.

The graph order is illustrated in Figure 6, while the geometric order is illustrated in Figure 7.

Lemma 10. *Let r be a realization of a graph G with an induced cycle C and intersectors D . If for every intersection of some $r(d)$ for $d \in D$ with some $r(c)$ for $c \in C$ we have that $r(d)$ also intersects the interior boundary curve b close to that intersection, then the geometric and graph order of the intersectors are the same.*

Proof. We first claim that every pair (c, d) occurs in both orders exactly once if $\{c, d\} \in E(G)$, and zero times otherwise: In the graph order this holds by definition. Furthermore, the assumption of this lemma guarantees that any pair (c, d) in the graph order of intersectors also occurs in the geometric order of intersectors. We thus only need to show that in the geometric order no pair occurs multiple times. A pair could only occur more than once if there is another pair (c', d') between these occurrences. However, since c is only crossed by d , and no neighbor of c in C is crossed by any intersector, Lemma 4 guarantees that no such pair (c', d') can occur between the two occurrences of (c, d) .

Next, we show that the pairs are also ordered the same way. Since for any two pairs $(c, d), (c', d')$ in the two vertices c, c' are distinct and non-adjacent, Lemma 4 guarantees that the geometric order of the intersectors must respect the ordering of the c_i along the cycle. Similarly, the graph order of intersectors must respect this ordering by definition. Thus, the two orders are the same. \square

Note that the definition of the geometric order and Lemma 10 also work for the exterior boundary curve instead of the interior one.

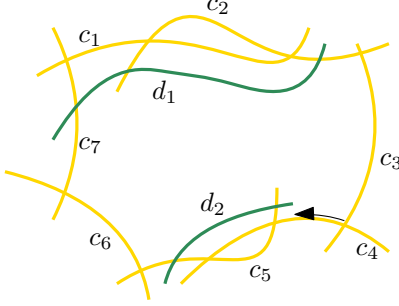


Figure 7: A realization of an induced cycle C (yellow Jordan arcs) and intersectors of C (green Jordan arcs). The cyclic order o_r along the interior boundary curve as used in Lemma 10 is here $(c_5, d_2), (c_7, d_1), (c_2, d_1)$, when starting at the bottom right and going clockwise (see black arrow).

3 $\exists\mathbb{R}$ -Membership

In this section, we show the $\exists\mathbb{R}$ -membership parts of Theorems 1 and 2.

There are two standard ways to establish $\exists\mathbb{R}$ -membership. The naive way is to encode the problem at hand as an ETR-formula. The second method describes a real witness and a real verification algorithm, similar as to how one can prove NP-membership using a combinatorial verification algorithm. We describe both approaches.

We describe the naive approach only for unit segments, however a similar technique also easily works for polylines. Let $G = (V, E)$ be a graph. For each vertex $v \in V$, we use four variables $\llbracket v1 \rrbracket, \llbracket v2 \rrbracket, \llbracket v3 \rrbracket, \llbracket v4 \rrbracket$ meant to describe the coordinates of the endpoints of a unit segment realizing v . We can construct ETR-formulas UNIT and INTERSECTION that test whether a segment has unit length and whether two segments are intersecting, respectively. The formula φ consists of the three parts:

$$\bigwedge_{v \in V} \text{UNIT}(\llbracket v1 \rrbracket, \llbracket v2 \rrbracket, \llbracket v3 \rrbracket, \llbracket v4 \rrbracket),$$

$$\bigwedge_{uv \in E} \text{INTERSECTION}(\llbracket u1 \rrbracket, \llbracket u2 \rrbracket, \llbracket u3 \rrbracket, \llbracket u4 \rrbracket, \llbracket v1 \rrbracket, \llbracket v2 \rrbracket, \llbracket v3 \rrbracket, \llbracket v4 \rrbracket),$$

and

$$\bigwedge_{uv \notin E} \neg \text{INTERSECTION}(\llbracket u1 \rrbracket, \llbracket u2 \rrbracket, \llbracket u3 \rrbracket, \llbracket u4 \rrbracket, \llbracket v1 \rrbracket, \llbracket v2 \rrbracket, \llbracket v3 \rrbracket, \llbracket v4 \rrbracket).$$

The UNIT formula can be constructed using the formula for the Euclidean norm. To construct the INTERSECTION formula it is possible to use the orientation test: The orientation test formula checks whether a given ordered triple of points is oriented clockwise, or counter-clockwise. Using multiple orientation tests on the endpoints of two segments one can determine whether the segments intersect. The orientation test itself can be constructed using a standard determinant test. This finishes the description of the formula φ and establishes $\exists\mathbb{R}$ -membership.

Although all of these formulas are straightforward to describe, things get a bit lengthy (especially in the case of polylines) and we do hide some details about the precise polynomials. We therefore also wish to describe the second approach using real witnesses and verification algorithms. To use this approach we need to first introduce a different characterization of the complexity class $\exists\mathbb{R}$. Namely, an algorithmic problem is in $\exists\mathbb{R}$ if and only if we can provide a *real verification algorithm* [19]. A real verification algorithm A for a problem P , takes as input an instance I of P and a polynomial-size real-valued witness w . A must have the following properties: In case that I is a yes-instance, there exists a w such that $A(I, w)$ returns yes. In case that I is a no-instance $A(I, w)$ will return no, for all possible w . Note that this is reminiscent of the definition of NP using a verifier algorithm. There are two subtle differences: The first one is that w is allowed

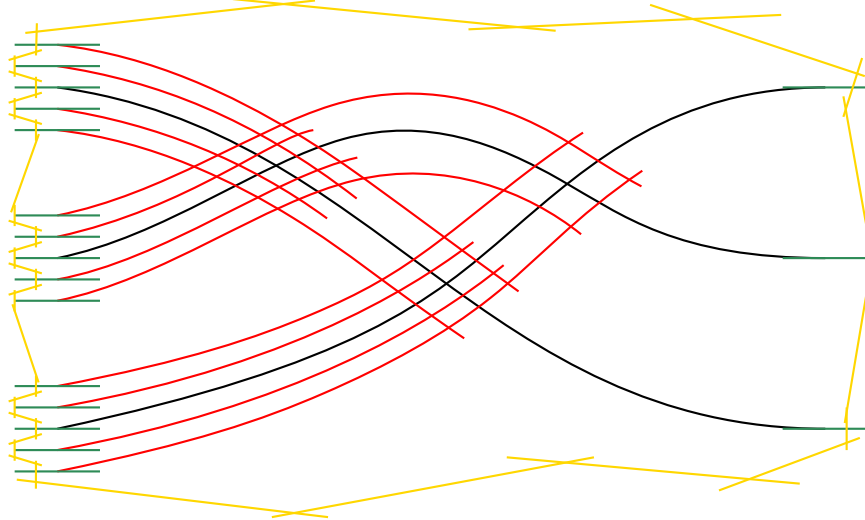


Figure 8: A pseudoline arrangement (black), enhanced with probes (red), connectors (green) and a cycle (yellow).

to contain *real* numbers and discrete values, not only bits. The second difference is that A runs on the *real RAM* instead of a Turing machine. This is required since a Turing machine is not capable of dealing with real numbers. It is important to note that I itself does not contain any real numbers. We refer to Erickson, Hoog, and Miltzow [19] for a detailed definition of the real RAM.

Given this alternative characterization of $\exists\mathbb{R}$, it is now very easy to establish $\exists\mathbb{R}$ -membership of UNIT RECOGNITION and k -POLYLINE RECOGNITION: We merely need to describe the witness and the verification algorithm. The witness is a description of the coordinates of the unit segments (or polylines, respectively) realizing the given graph. The verification algorithm merely checks that the correct realizations of vertices intersect one another, and in the case of unit segments, also checks that all segments have the correct length. Note that we sweep many details of the algorithm under the carpet. However, algorithms are much more versatile than formulas and it is a well-established fact that algorithms are capable of all types of elementary operations needed to perform this verification.

4 $\exists\mathbb{R}$ -Hardness

In this section we prove $\exists\mathbb{R}$ -hardness, first for unit segments, then for polylines. The reduction for polylines builds upon the reduction for unit segments and we will only highlight the differences.

4.1 $\exists\mathbb{R}$ -Hardness for Unit Segments

We show $\exists\mathbb{R}$ -hardness of UNIT RECOGNITION by a reduction from STRETCHABILITY.

We present the reduction in three steps. First, we show how we construct a graph G from a pseudoline arrangement \mathcal{A} . Then we show completeness, i.e., we show that if \mathcal{A} is stretchable then G can be represented using unit segments. At last, we will show soundness, i.e., we show that if G can be represented using unit segments then \mathcal{A} is stretchable. For this part we will use the lemmas from Section 2.

Construction. Given a pseudoline arrangement \mathcal{A} of n pseudolines, we construct a graph G by enhancing the arrangement \mathcal{A} with additional Jordan arcs. Then we define G to be the intersection graph of the pseudolines (which are also Jordan arcs) and all our added arcs. See Figure 8 for an illustration.

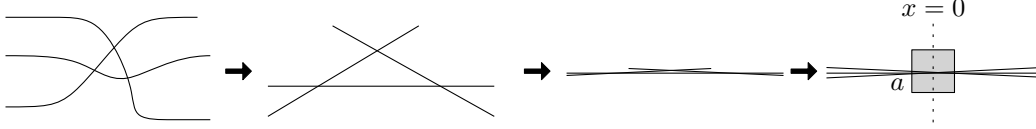


Figure 9: Left to Right: pseudoline arrangement \mathcal{A} ; stretched line arrangement \mathcal{L} , squeezed line arrangement to have small slopes; squeezed to have all intersections in a box of side length a .

First, we add so called *probes* to our pseudolines. A probe of *depth* k of pseudoline ℓ is a Jordan arc which starts on the left vertical line, and follows ℓ closely through the arrangement until it has intersected k other pseudolines (and all probes of that other pseudoline which reach that intersection). For each pseudoline ℓ , we add $2(n-1)$ probes: Above and below ℓ we add one probe each for each depth $1 \leq k \leq n-1$. This gives a total of $p := 2n(n-1)$ probes in the graph G . The probes are sorted according to their depth, with the probes of smallest depth situated closest to ℓ . Note that so far we have added $n+p$ arcs. We now create *connectors* for each probe and pseudoline. A connector is a short Jordan arc added to the left and/or right end of another arc. For probes, we only add connectors at the left end, while pseudolines get connectors at both ends. Note that we thus add $d := 2n+p$ connectors. Finally, we add Jordan arcs forming cycle of length $2d+6$ around the current arrangement. At the left and right end, this cycle is placed in such a way that every second arc of the cycle intersects a connector, in the correct order. The eight additional arcs of the cycle are used to connect the left and right side of the cycle, using four arcs on the top and four arcs on the bottom.

We denote the collection of all those arcs (including the pseudolines) as the *enhanced pseudoline arrangement*. The graph G is given by the intersection graph of this arrangement. Note that G could also be described purely combinatorially, and it can be constructed in polynomial time.

Completeness. This paragraph is dedicated to show that if \mathcal{A} is stretchable then G is realizable by unit segments. We assume that \mathcal{A} is stretchable and show how to place each unit segment realizing G . We denote the segments representing the pseudolines, probes, connectors, and the cycle by *important segments*, *probe segments*, *connector segments* and *cycle segments*.

\mathcal{A} being stretchable implies that there exists a combinatorially equivalent line arrangement \mathcal{L} . The arrangement \mathcal{L} can be compressed (scaled down along the vertical axis) such that the slopes of all lines lie in some small interval $[-a, a]$, for say $a = 1/20$. Additionally, we can move and scale \mathcal{L} even more to ensure that all intersection points lie in the square $[-a, a]^2$. See Figure 9 for an illustration.

We now truncate all the lines of \mathcal{L} to get unit segments that have the same intersection pattern as the corresponding lines. The truncation is performed symmetrically around the vertical line $x = 0$. The resulting unit segments are our *important segments*.

For the next step we consider an important segment s and construct its corresponding *probe segments*. We place all probe segments parallel to its important segment s , with sufficiently small distance to s and to each other. The probe segments are placed as far towards the left as possible, while still reaching the intersections of important segments they need to reach. Since by construction of the scaled line arrangement \mathcal{L} all intersections lie within $[-a, a]^2$ and the probes can only go until there, each probe segment and its corresponding important segment are almost collinear but shifted by roughly 0.5 longitudinally. See Figure 10 for an illustration of the placement of the probe segments.

Next we need to describe the placement of the *connector segments*. Note that on the right side, we only have important segments. We add all the connector segments in such a way that they lie on the same line as the important/probe segment they attach to. The connectors can overlap with the segments they attach to for a large part, since the first intersection of any probe and important segments only occurs in the square $[-a, a]^2$. This allows us to place our connectors such that all connectors on the left (right) side of the drawing end at the same left (right) x -coordinate.

Finally, we can draw the cycle segments. For this, we simply make a sawtooth pattern on the left and on the right. See Figure 11 for an illustration. In our sawtooth pattern, every second cycle segment is horizontal,

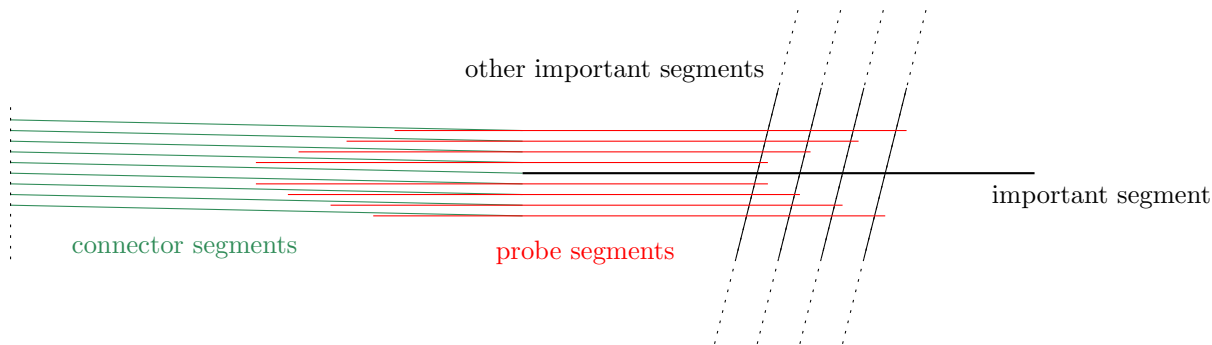


Figure 10: Given \mathcal{L} , we can construct the probe segments and connector segments using unit segments. The connector segments are illustrated slightly tilted, for the purpose of readability. They should be parallel to the segment that they connect to.

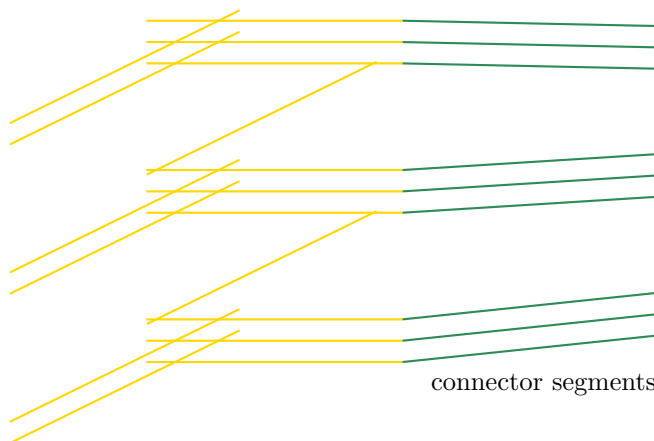


Figure 11: We can place all the cycle segments correctly on the left and on the right using a simple sawtooth pattern.

and all other cycle segments are parallel. The horizontal segments attach to the connector segments. As the important segments all have a very small slope, we can never run into a situation where two horizontal cycle segments would be too far away from each other to be connected. We connect the left and right sawtooth patterns to close the cycle, using our eight additional cycle segments.

Soundness. This paragraph is dedicated to show that G being realizable using unit segments implies that \mathcal{A} is stretchable. We thus assume there exists a realization r of G by unit segments. Similarly to the last paragraph, we denote the vertices representing the pseudolines, probes, connectors, and the cycle by *important vertices* I , *probe vertices* P , *connector vertices* D , and *cycle vertices* C .

Note that C forms an induced cycle in G and D are connectors of C as in Definition 5 (thus motivating the name). By Lemma 6, $r(C)$ splits the plane into two cells, and $r(I \cup P)$ is contained completely in one of these two cells. Without loss of generality, we assume $r(I \cup P)$ is contained in the inner (bounded) cell, however all following arguments would also work with the outer cell. Thus, every segment in $r(D)$ intersects the cycle $r(C)$ from the inside, i.e., it intersects the interior boundary curve. We can thus apply Lemma 10, and get that $r(D)$ is ordered along the interior boundary curve of C in the same way (up to cyclic shift and reflection) as it is in our enhanced pseudoline arrangement as described in the construction of G . Specifically, we know that the important segments and probe segments are ordered as in our enhanced pseudoline arrangement, see Figure 8.

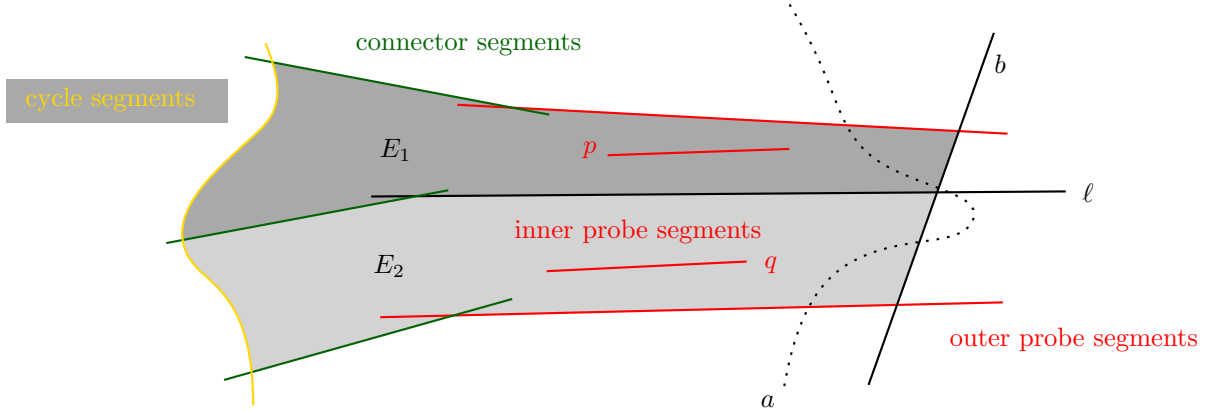


Figure 12: Suppose for the purpose of contradiction that ℓ intersects a after b . Then the important segment a can only intersect one of the cells E_1 and E_2 . However, E_1 and E_2 contain a segment that a intersects in its interior. A contradiction.

We now claim that the arrangement of the important segments $r(I)$ is combinatorially equivalent to \mathcal{A} . Thus, extending these segments to lines yields a combinatorially equivalent line arrangement. To see this, we pick three pseudolines ℓ , a , and b , such that ℓ intersects a before b when going from left to right in \mathcal{A} . We prove that in our realization r , the unit segment $r(\ell)$ must also intersect the segment $r(a)$ before $r(b)$. The orientation of the unit segments $r(\ell)$ is determined as follows: $r(\ell)$ intersects two connectors d_1, d_2 that intersect $r(C)$ in two different unit segments. One of these unit segments c (intersected by connector d_1) is two segments away from a unit segment that intersects a connector that intersects a probe of ℓ . We then orient $r(\ell)$ in the direction such that the intersection with d_1 occurs before the intersection with d_2 . Note that since d_1 and d_2 do not intersect, this order is uniquely defined. Furthermore, simply rotating or mirroring the representation does not change this order.

Now, let us suppose for the purpose of a contradiction that $r(\ell)$ intersects $r(b)$ before $r(a)$. Let us consider the following curves:

- the outermost probe segments of ℓ ,
- their connector segments,
- a part of the interior boundary curve of the cycle segments, and
- the important segment b .

Since these segments form an induced cycle in the graph G , these segments bound a cell E . In this cycle, ℓ and its connector segment d_1 (the one attaching to the cycle between the probes) form a chord, thus their representations split E into two parts E_1, E_2 . See Figure 12 for an illustration. Note that $r(\ell)$ must be oriented from the end contained in E towards the other end, since the intersection of $r(d_1)$ and $r(\ell)$ lies in E , but the intersection of $r(d_2)$ and $r(\ell)$ cannot. Since we assume that $r(\ell)$ intersects $r(b)$ before $r(a)$, the intersection of $r(a)$ and $r(\ell)$ thus lies outside of E .

We consider the two probe segments p, q of ℓ , which correspond to the intersection with a . These segments are attached to the cycle between the outermost probe segments of ℓ , and ℓ itself. Furthermore, p and q are not intersecting any segment bounding E_1 or E_2 , in particular not $r(b)$. Thus the probe segments p and q are both completely contained in the interior of E_1 and E_2 , respectively. However, $r(a)$ can intersect the interior of only one of E_1 and E_2 , but not both, since $r(a)$ and $r(\ell)$ are line segments and their single intersection is assumed to lie outside of E . Since both p and q must intersect $r(a)$, we arrive at the desired contradiction. We thus conclude that \mathcal{A} is stretchable, finishing the proof of Theorem 1.

4.2 $\exists\mathbb{R}$ -Hardness for Polyline

We study polylines with k bends and we assume $k > 0$ is a fixed constant. Specifically, we will show that k -POLYLINE RECOGNITION is $\exists\mathbb{R}$ -complete.

The proof for polylines works very similarly to the proof for unit segments. Since the family of polylines is a strict superset of the family of unit segments, most of our additional work is on the soundness of the proof. To be able to ensure soundness, our construction of the graph G will make sure that the polylines realizing the pseudolines cannot contain any bends in some region encoding the pseudoline arrangement \mathcal{A} in any realization of G . With this property, the argument for soundness (as in the proof of Theorem 1) will carry over straightforwardly.

Construction. As in the construction for unit segments, we enhance our arrangement \mathcal{A} of n pseudolines using additional Jordan arcs, and let our graph G be the intersection graph of the enhanced arrangement.

To create the enhanced arrangement, we first create the *frame*: We create $2k + 2$ vertical chains of $c \cdot n^2$ line segments, for some c sufficiently large. We call the set of segments forming the i -th such vertical chain C_i , for $i = 1 \dots, 2k + 2$, numbered from left to right. We then connect these vertical chains by two horizontal chains, one at the top and one at the bottom. All of the segments forming these chains are called *frame segments*, and they together bound $2k + 1$ bounded cells. We call the leftmost such cell the *canvas*.

We now place our pseudoline arrangement in the canvas. For every pseudoline p , we first introduce a parallel *twin* pseudoline p' closely below p . Since \mathcal{A} is assumed to be simple, every pseudoline has the same intersection pattern with all the other pseudolines as its twin.

Next, we introduce probes from the left of our pseudoline arrangement, as we did in the proof of Theorem 1. Note that every pair of twin pseudolines shares one set of probes. We attach the probes as well as the pseudolines and their twins to C_1 using connector arcs, making sure that each connector intersects a unique arc of C_1 , and that none of these arcs are intersecting.

We now want to weave the pseudolines and their twins through C_2, \dots, C_{2k+1} . To do this, we split each set C_i into n lanes $L_{i,1}, \dots, L_{i,n}$: A lane is a set of three consecutive arcs of C_i . The lanes $L_{i,j}$ are pairwise disjoint, and no arc of any lane $L_{i,j}$ may intersect an arc of a lane $L_{i,j'}$. Note that the lanes are ordered along C_i with $L_{i,1}$ being the topmost lane and $L_{i,n}$ the bottommost. We also number our pseudolines from p_1 to p_n by their order at the right end of the arrangement, from top to bottom. For every pseudoline p_j , we now extend p_j and its twin p'_j as follows: For each even i , p_j goes through the top arc of the lane $L_{i,j}$, and for each odd i , it goes through the bottom arc of $L_{i,j}$. The twin p'_j is extended in the opposite way, going through the top arc of $L_{i,j}$ if i is odd, and through the bottom arc for i even.

At the very end, at C_{2k+2} , we use a connector arc to attach each pseudoline p_j to the top element of $L_{2k+2,j}$, and its twin p'_j to the bottom element. This whole construction is illustrated in Figure 13.

Completeness. Given a line arrangement \mathcal{L} combinatorially equivalent to \mathcal{A} , we want to show that G is realizable by polylines. It is easy to see that given \mathcal{L} , the canvas and its contents (probes, connectors, pseudolines, and pseudoline twins) can be realized even with line segments (i.e., using 0 bends), as we argued already in the proof of Theorem 1. The only remaining difficulty is to argue that the extended pseudolines weaving through C_2, \dots, C_{2k+1} and finally attaching to C_{2k+2} can be realized using at most k bends per pseudoline and twin.

Since between every two lanes $L_{i,j}$ and $L_{i,j+1}$ there is at least one polyline that is not part of any lane, we can make the distance between these lanes arbitrarily large. We thus only need to show that for a single pair p_j, p'_j , we can weave the polylines through $L_{2,j}, \dots, L_{2k+2,j}$. The construction is illustrated in Figure 14: The two polylines exit the canvas in parallel. Then, going from left to right, we first use all bends of p_j , leaving its twin straight. After having gone through $L_{2,j}, \dots, L_{k+2,j}$, we leave p_j straight and use the bends of the twin to go through the remaining lanes $L_{k+3,j}, \dots, L_{2k+1,j}$ and to have the right ordering to attach to C_{2k+2} using a connector.

Soundness. Given a realization of the graph G with polylines, we want to argue that \mathcal{A} is stretchable. To achieve this, we first prove some structural lemmas about all realizations of G .

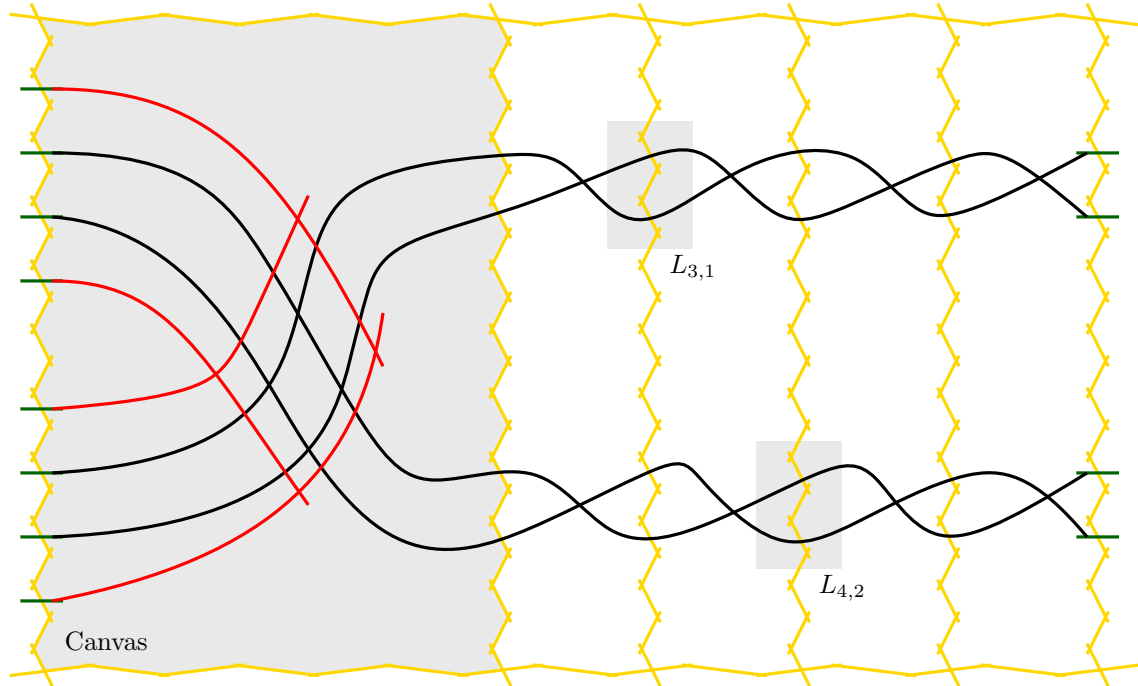


Figure 13: A doubled pseudoline arrangement (black) enhanced for $k = 2$ with probes (red), connectors (green) and the frame (yellow). The shaded region is the canvas.

Lemma 11. *In every realization $r(G)$, the polylines realizing all probes, pseudolines, and internal vertices of C_2, \dots, C_{2k+1} must lie on the same side of the boundary curves given by the cycle bounded by C_1 and C_{2k+2} .*

Proof. The induced subgraph of this set of vertices is connected. The connectors attaching them to the outermost cycle together with the outermost segments of C_2, \dots, C_{2k+1} thus form a set of connectors as defined in Definition 5. Thus the lemma follows from Lemma 6 immediately. \square

We again assume without loss of generality that all these elements lie in the interior (bounded) cell bounded by the outermost cycle. We now know that for every pair of consecutive chords C_i, C_{i+1} , any element $r(v)$ that intersects the cycle bounded by these chords must also intersect the interior boundary curve of that cycle. We can thus apply Lemma 10 to these cycles to enforce the ordering of crossings along them. From this, similarly as in the proof of soundness for unit segments (note that that proof did not use

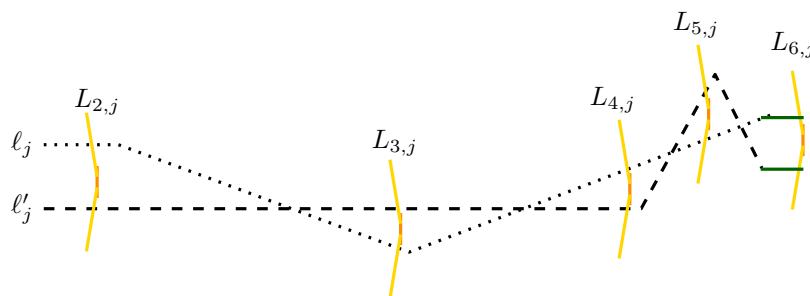


Figure 14: Weaving the two copies (dashed and dotted) of a pseudoline through the $2k$ chords with at most k bends per polyline. Illustrated for $k = 2$.

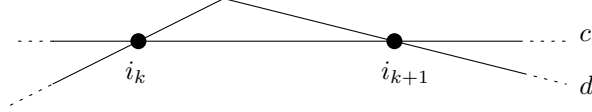


Figure 15: It is easy to see that the two polylines must have at least one bend between the two intersection points.

the fact that the line segments were unit), we wish to get that the arrangement obtained by picking either $r(p_j)$ or $r(p'_j)$ for every $j \in [n]$ restricted to the canvas has the same combinatorial structure as \mathcal{A} . However, for this it remains to be shown that at least one of $r(p_j)$ and $r(p'_j)$ must be a straight line within the canvas. To show this, we first show that the two polylines $r(p_j)$ and $r(p'_j)$ must cross often:

Lemma 12. *Within the cell enclosed by the interior boundary curve of the cycle enclosed by two consecutive sets C_i, C_{i+1} , for $2 \leq i \leq 2k + 1$, each $r(p_j)$ and $r(p'_j)$ must cross.*

Proof. By Lemma 10 and by construction of G , the geometric order of $r(p_j)$ and $r(p'_j)$ along the interior boundary curve of the cell bounded by C_i and C_{i+1} must be alternating (non-nesting). Thus, the two polylines $r(p_j)$ and $r(p'_j)$ must cross within this cell. \square

Lemma 13. *Let c, d be two polylines that intersect in exactly t points i_1, \dots, i_t , with both polylines visiting the intersection points in this order. Then c and d have at least $t - 1$ bends in total between the first and the last intersection.*

Proof. It is easy to see that there must be at least one bend between any two consecutive intersection points. See Figure 15 for an illustration. \square

We now finally get our desired lemma:

Lemma 14. *At least one of $r(p_j)$ and $r(p'_j)$ is a straight line within the canvas.*

Proof. The two twin polylines must have at least $2k$ intersection points that occur in the same order along the two twin: to find these points, we pick one per cell enclosed between C_i, C_{i+1} for $2 \leq i \leq 2k + 1$ as guaranteed by Lemma 12. Thus by Lemma 13, the two polylines have at least $2k - 1$ bends in total outside of the canvas. This implies that at least one of them is straight inside the canvas. \square

We conclude that \mathcal{A} must be stretchable, finishing the proof of Theorem 2.

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