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Chapter 5

On Multi-Time Correlations in Stochastic Mechanics



Maaneli Derakhshani and Guido Bacciagaluppi

Abstract We address a long-standing criticism of the stochastic mechanics approach to quantum theory by one of its pioneers, Edward Nelson: multi-time correlations in stochastic mechanics differ from those in textbook quantum theory. We elaborate upon an answer to this criticism by Blanchard et al. (Phys Rev D 34(12):3732–3738, 1986), who showed that if the (derived) wave function in stochastic mechanics is assumed to collapse to a delta function in a position measurement, the collapse will change the stochastic process for the particles (because the stochastic process depends on derivatives of the wave function), and the resulting multi-time correlations will agree with those in textbook quantum theory. We show that this assumption can be made rigorous through the tool of ‘effective collapse’ familiar from pilot-wave theories, and we illustrate this with an example involving the double-slit experiment. Hence one of the major lingering objections to stochastic mechanics is dissolved. We finally show that in the case of multi-time correlations between multiple particles, effective collapse implies nonlocal influences between particles.

Keywords Stochastic mechanics · Multi-time correlations · Effective collapse

5.1 Introduction

Stochastic mechanics is an approach to quantum theory that aims to recover it from an underlying stochastic process in configuration space. As developed by Nelson (1966, 1985), and briefly described in Sect. 5.2 for the N -particle case,

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such a recovery starts from a time-reversible description of a diffusion process in configuration space. By then imposing a number of (time-symmetric) dynamical conditions on the process, most elegantly the variational principle due to Yasue (1981), one obtains the Madelung equations for two real functions R and S , which can be combined into a Schrödinger equation for $\psi = e^{\frac{i}{\hbar}(R+iS)}$.

The resulting theory has a number of similarities with de Broglie and Bohm's pilot-wave theory. Indeed, particle trajectories in Nelson's stochastic mechanics can be intuitively thought of as de Broglie–Bohm trajectories with a superimposed white noise (although in pilot-wave theory ψ is interpreted as a fundamental quantity). This means, of course, that stochastic mechanics shares the same non-local features as pilot-wave theory (one of the reasons why Nelson himself eventually lost interest in his theory). As discussed elsewhere (Bacciagaluppi 2005), however, stochastic mechanics has a number of potential advantages over pilot-wave theory. The main reason why it is not as popular as de Broglie–Bohm theory (or other fundamental approaches to quantum theory such as spontaneous collapse or many-worlds), we surmise, is that it is perceived as suffering from two main technical problems not shared by other better-known approaches.

These are:

- (a) The problem, raised by Wallstrom (1994), that the Madelung equations are only equivalent to the Schrödinger equation if one imposes a seemingly ad hoc quantisation condition on their solutions.
- (b) The problem, raised by Nelson himself (1986), that the theory appears to fail to recover the multi-time correlations predicted by quantum theory.

For different reasons, however, neither of these problems are decisive. Wallstrom's problem proved recalcitrant for many years, but after a number of arguably only partially convincing attempts,¹ it has now found what appears to us to be a natural resolution, based on combining Nelson's strategy with de Broglie's original strategy for deriving the quantisation conditions (Derakhshani 2017). And, as we argue in this paper, Nelson's problem of multi-time correlations is a red herring.

The problem of multi-time correlations (which we describe in more detail in Sect. 5.3) can be expressed very intuitively in terms of series of measurements. In standard quantum theory, in order to calculate the correlations between the position of a particle at multiple times, one considers a series of position measurements, which localise the particle at the given times. Straightforward application of the Born rule provides the desired result. In Nelson's mechanics, the particle has a well-defined position at all times, and the stochastic process also naturally determines the correlations in position at multiple times. The quantum mechanical calculation and the stochastic mechanical calculation, however, lead to different results, as spelled out by Nelson (1986).

At the root of the problem lies the fact that in the quantum mechanical calculation at each measurement one substitutes a new collapsed wave function for the original

¹ The most interesting probably being Schmelzer (2011).

unitarily evolved wave function, while the wave function that is derived in Nelson's mechanics from the underlying stochastic process always follows the unitary Schrödinger evolution. Indeed, as already pointed out by Blanchard et al. (1986) (and summarised below in Sect. 5.4), if the corresponding substitution is made also in the stochastic mechanical calculation, the results will coincide. The obvious question, however, is *why on earth* should one feel licensed to invoke collapse in Nelson's theory where the wave function is not even a fundamental object?

The answer, as we shall argue in detail in Sect. 5.5, lies just below the surface. Since the current velocity $\mathbf{v}$ in stochastic mechanics has the same form as the particle velocity in de Broglie–Bohm theory, many of the results and techniques developed in the context of pilot-wave theory can be straightforwardly imported into stochastic mechanics, even though in the latter theory the wave function is not considered a fundamental quantity. One example, transferred to stochastic mechanics in Bacciagaluppi (2012), is the notion of ('sub-quantum') disequilibrium and consequences thereof, i.e. the idea that frequencies in an actual sample could differ from the theoretical probability distribution. The pilot-wave notion we focus on in this paper, however, is that of *effective collapse*, which is the tool developed by Bohm (1952) to describe in pilot-wave theory measurements of quantities other than position. The key idea is that one must include the apparatus into the pilot-wave theoretic description. Since the trajectory of one system depends on the actual position of other systems with which it is entangled, including information about the position of such other systems changes the way one calculates the trajectory of a system. And in measurement situations, including information about the measurement result means that one can calculate trajectories as if the wave function of the system had collapsed. As we shall discuss, such effective collapse can be transferred also to the setting of stochastic mechanics, and as we have suggested previously (Bacciagaluppi 2012, p. 5, fn. 8), it is the missing ingredient that allows one to solve Nelson's problem of multi-time correlations.²

We conclude that the common perception of stochastic mechanics as suffering from major technical problems is mistaken.

² The notion of effective collapse has been discussed in stochastic mechanics by Goldstein (1987) as well as by Blanchard et al. (1991), but does not seem to have found wide application. Peruzzi and Rimini (1996) give it a unified treatment in pilot-wave theory, stochastic mechanics and variants thereof, and Bacciagaluppi (2003) makes essential use of it in discussing identical particles in pilot-wave theory and stochastic mechanics. Blanchard et al. (1991, p. 162) clearly realise its relevance to the problem of multi-time correlations (since they reference Blanchard et al. (1986)), but they do not spell out the connection explicitly. Nelson, to the best of our knowledge, never discussed or recognised the notion of effective collapse in his writings on stochastic mechanics; if he was unaware of the notion, this could help explain why he thought there was a fundamental discrepancy between stochastic mechanics and quantum mechanics for multi-time correlations.

5.2 Review of Nelson–Yasue Stochastic Mechanics

The N -particle version of Nelson–Yasue stochastic mechanics (NYSM) begins with the hypothesis that 3-D space is pervaded by a homogeneous and isotropic ether with classical stochastic fluctuations that impart a frictionless, conservative diffusion process to a point particle of mass m and charge e immersed within the ether. Accordingly, for N point particles of masses m_i and charges e_i immersed in the ether, each particle will in general have its position 3-vector $\mathbf{q}_i(t)$ constantly undergoing diffusive motion with drift, as modelled by the first-order forward stochastic differential equations

$$d\mathbf{q}_i(t) = \mathbf{b}_i(q(t), t)dt + d\mathbf{W}_i(t). \quad (5.1)$$

Here $q(t) := \{\mathbf{q}_1(t), \mathbf{q}_2(t), \dots, \mathbf{q}_N(t)\} \in \mathbb{R}^{3N}$, $\mathbf{b}_i(q(t), t)$ is the deterministic mean forward drift velocity of the i th particle (which in general may be a function of the positions of all the other particles, such as in the case of particles interacting with each other gravitationally and/or electrostatically), and $\mathbf{W}_i(t)$ is the Wiener process modeling the i th particle's interaction with the ether fluctuations. (The 'mean' here refers to averaging over the Wiener process, in the sense of the conditional expectation at time t .)

The Wiener increments $d\mathbf{W}_i(t)$ are assumed to be Gaussian with zero mean, independent of $d\mathbf{q}_i(s)$ for $s \leq t$, and with variance

$$E_t [dW_{in}(t)dW_{im}(t)] = 2\nu_i \delta_{nm}dt, \quad (5.2)$$

where E_t denotes the conditional expectation at time t . We then hypothesise that the magnitudes of the diffusion coefficients ν_i are given by

$$\nu_i = \frac{\hbar}{2m_i}. \quad (5.3)$$

In addition to (5.1), we also have the backward stochastic differential equations

$$d\mathbf{q}_i(t) = \mathbf{b}_{i*}(q(t), t)dt + d\mathbf{W}_{i*}(t), \quad (5.4)$$

where $\mathbf{b}_{i*}(q(t), t)$ are the mean backward drift velocities, and $d\mathbf{W}_{i*}(t)$ are the backward Wiener processes. The $d\mathbf{W}_{i*}(t)$ have all the properties of $d\mathbf{W}_i(t)$ except that they are independent of the $d\mathbf{q}_i(s)$ for $s \geq t$. With these conditions on $d\mathbf{W}_i(t)$ and $d\mathbf{W}_{i*}(t)$, Eqs. (5.1) and (5.4) respectively define forward and backward Markov processes for N particles on $\mathbb{R}^3$ (or, mathematically equivalently, for a single particle on $\mathbb{R}^{3N}$).

Associated to the trajectories $\mathbf{q}_i(t)$ is the N -particle probability density $\rho(q, t) := n(q, t)/N$ where $n(q, t)$ is the number of particles per unit volume. Corresponding to (5.1) and (5.4), then, are the N -particle forward and backward

Fokker-Planck equations

$$\frac{\partial \rho(q, t)}{\partial t} = - \sum_{i=1}^N \nabla_i \cdot [\mathbf{b}_i(q, t) \rho(q, t)] + \sum_{i=1}^N \frac{\hbar}{2m_i} \nabla_i^2 \rho(q, t), \quad (5.5)$$

and

$$\frac{\partial \rho(q, t)}{\partial t} = - \sum_{i=1}^N \nabla_i \cdot [\mathbf{b}_{i*}(q, t) \rho(q, t)] - \sum_{i=1}^N \frac{\hbar}{2m_i} \nabla_i^2 \rho(q, t), \quad (5.6)$$

where we assume that the solutions $\rho(q, t)$ in each time direction satisfy the normalisation condition

$$\int_{\mathbb{R}^{3N}} \rho_0(q) d^{3N}q = 1. \quad (5.7)$$

Up to this point, (5.5) and (5.6) correspond to independent diffusion processes in opposite time directions.³ To fix the diffusion process uniquely for both time directions, we must constrain the diffusion process to simultaneous solutions of (5.5) and (5.6).

Note that the sum of (5.5) and (5.6) yields the N -particle continuity equation

$$\frac{\partial \rho(q, t)}{\partial t} = - \sum_{i=1}^N \nabla_i \cdot [\mathbf{v}_i(q, t) \rho(q, t)], \quad (5.8)$$

where

$$\mathbf{v}_i(q, t) := \frac{1}{2} (\mathbf{b}_i(q, t) + \mathbf{b}_{i*}(q, t)) \quad (5.9)$$

is the current velocity field of the i th particle. We shall also require that $\mathbf{v}_i(q, t)$ is equal to the gradient of a scalar potential $S(q, t)$ (since, if we allowed $\mathbf{v}_i(q, t)$ a non-zero curl, then the time-reversal operation would change the orientation of the curl, thus distinguishing time directions).⁴ And for particles classically interacting with an external vector potential $\mathbf{A}_i^{ext} := \mathbf{A}^{ext}(\mathbf{q}_i, t)$, the current velocities get modified

³ In fact, given all possible solutions to (5.1), one can define as many forward processes as there are possible initial distributions satisfying (5.5); likewise, given all possible solutions to (5.4), one can define as many backward processes as there are possible ‘initial’ distributions satisfying (5.6). Consequently, the forward and backward processes are both underdetermined, and neither (5.1) nor (5.4) has a well-defined time-reversal.

⁴ Alternatively, assuming that $\mathbf{v}_i$ is a gradient ensures satisfaction of the symmetry condition $D\mathbf{b}_i = D_*\mathbf{b}_{i*}$ (cf. (5.15) and (5.16) below) (de la Peña and Cetto 1982, eq. (16); Bacciagaluppi 2012, Appendix).

by the usual expression

$$\mathbf{v}_i(q, t) = \frac{\nabla_i S(q, t)}{m_i} - \frac{e_i}{m_i c} \mathbf{A}_i^{ext}. \quad (5.10)$$

So (5.8) becomes

$$\frac{\partial \rho(q, t)}{\partial t} = - \sum_{i=1}^N \nabla_i \cdot \left[\left(\frac{\nabla_i S(q, t)}{m_i} - \frac{e_i}{m_i c} \mathbf{A}_i^{ext} \right) \rho(q, t) \right], \quad (5.11)$$

which is now a time-reversal invariant evolution equation for ρ .

The function S is an N -particle velocity potential, defined here as a field over the possible positions of the particles (hence the dependence of S on the generalised coordinates $\mathbf{q}_i$), and generates different possible initial irrotational velocities for the particles via (5.10). No assumptions are made at this level as to whether or not S can be written as a sum of single-particle velocity potentials. Rather, this will depend on the initial conditions and constraints specified for a system of N Nelsonian particles, as well as the dynamics we obtain for S .

Note also that subtracting (5.5) from (5.6) yields the equality on the right-hand side of

$$\mathbf{u}_i(q, t) := \frac{1}{2} [\mathbf{b}_i(q, t) - \mathbf{b}_{i*}(q, t)] = \frac{\hbar}{2m_i} \frac{\nabla_i \rho(q, t)}{\rho(q, t)}, \quad (5.12)$$

where $\mathbf{u}_i(q, t)$ is the osmotic velocity field of the i th particle. From (5.10) and (5.12), we then have $\mathbf{b}_i = \mathbf{v}_i + \mathbf{u}_i$ and $\mathbf{b}_{i*} = \mathbf{v}_i - \mathbf{u}_i$, which when inserted back into (5.5) and (5.6), respectively, return (5.11). Thus ρ is fixed as the unique, single-time, ‘quantum equilibrium’ distribution for the solutions of (5.1) and (5.4), and evolves by (5.11).

To give physical meaning to the osmotic velocities, we postulate the presence of an external ‘osmotic’ potential which couples to the i th particle as $R(q(t), t) = \mu U(q(t), t)$ and imparts to it a momentum $\nabla_i R(q, t)|_{q=q(t)}$. This momentum then gets counter-balanced by the ether fluid’s osmotic impulse pressure, $(\hbar/2m_i) \nabla_i \ln[n(q, t)]|_{q=q(t)}$, leading to the equilibrium condition $\nabla_i R/m_i = (\hbar/2m_i) \nabla_i \rho/\rho$ (using $\rho = n/N$), hence $\rho = e^{2R/\hbar}$ for all times.

In order to formulate the second-order dynamics of the particles, we need to generalise to N particles Nelson’s (1966) construction of mean forward and backward derivatives. This generalisation is straightforwardly given by

$$D\mathbf{q}_i(t) = \lim_{\Delta t \rightarrow 0^+} E_t \left[\frac{q_i(t + \Delta t) - q_i(t)}{\Delta t} \right], \quad (5.13)$$

and

$$D_* \mathbf{q}_i(t) = \lim_{\Delta t \rightarrow 0^+} \mathbb{E}_t \left[\frac{q_i(t) - q_i(t - \Delta t)}{\Delta t} \right]. \quad (5.14)$$

By the Gaussianity of $d\mathbf{W}_i(t)$ and $d\mathbf{W}_{i*}(t)$, we obtain $D\mathbf{q}_i(t) = \mathbf{b}_i(q(t), t)$ and $D_*\mathbf{q}_i(t) = \mathbf{b}_{i*}(q(t), t)$. We note that $\mathbf{v}_i(q(t), t) = \frac{1}{2}(D + D_*)\mathbf{q}_i(t)$. The computation of $D\mathbf{b}_i(q(t), t)$ and $D_*\mathbf{b}_i(q(t), t)$ is given in the appendix and yields

$$D\mathbf{b}_i(q(t), t) = \left[\frac{\partial}{\partial t} + \sum_{i=1}^N \mathbf{b}_i(q(t), t) \cdot \nabla_i + \sum_{i=1}^N \frac{\hbar}{2m_i} \nabla_i^2 \right] \mathbf{b}_i(q(t), t), \quad (5.15)$$

$$D_*\mathbf{b}_{i*}(q(t), t) = \left[\frac{\partial}{\partial t} + \sum_{i=1}^N \mathbf{b}_{i*}(q(t), t) \cdot \nabla_i - \sum_{i=1}^N \frac{\hbar}{2m_i} \nabla_i^2 \right] \mathbf{b}_{i*}(q(t), t). \quad (5.16)$$

We now generalise the construction given by Yasue (1981) in the single-particle case. Using (5.15)–(5.16), and assuming that particle i also couples to an external electric potential, $\Phi_i^{ext} := \Phi^{ext}(\mathbf{q}_i(t), t)$, as well as to the other particles by the Coulomb interaction potential $\Phi_i^{int} := \frac{1}{2} \sum_{j=1}^{N(j \neq i)} \frac{e_j}{|\mathbf{q}_i(t) - \mathbf{q}_j(t)|}$, we can construct the N -particle generalisation of Yasue's ensemble-averaged, time-symmetric mean action:

$$\begin{aligned} J &= \mathbb{E} \left[\int_{t_I}^{t_F} \sum_{i=1}^N \left\{ \frac{1}{2} \left[\frac{1}{2} m_i \mathbf{b}_i^2 + \frac{1}{2} m_i \mathbf{b}_{i*}^2 \right] + \frac{e_i}{c} \mathbf{A}_i^{ext} \cdot \frac{1}{2} (D + D_*) \mathbf{q}_i(t) \right. \right. \\ &\quad \left. \left. - e_i \left[\Phi_i^{ext} + \Phi_i^{int} \right] \right\} dt \right] \\ &= \mathbb{E} \left[\int_{t_I}^{t_F} \sum_{i=1}^N \left\{ \frac{1}{2} m_i \mathbf{v}_i^2 + \frac{1}{2} m_i \mathbf{u}_i^2 + \frac{e_i}{c} \mathbf{A}_i^{ext} \cdot \mathbf{v}_i - e_i \left[\Phi_i^{ext} + \Phi_i^{int} \right] \right\} dt \right], \end{aligned} \quad (5.17)$$

where $\mathbb{E}[\dots]$ denotes the absolute expectation.⁵

Upon imposing the conservative diffusion constraint through the N -particle generalisation of Yasue's variational principle

$$J = \text{extremal}, \quad (5.18)$$

⁵ $\mathbb{E}[\dots] := \int \rho(q, t) [\dots] d^{3N}q$.

a straightforward computation (Derakhshani 2017, Appendix 7.1) shows that (5.18) implies

$$\begin{aligned} & \sum_{i=1}^N \frac{m_i}{2} [D_* D + D D_*] \mathbf{q}_i(t) \\ &= \sum_{i=1}^N e_i \left[-\frac{1}{c} \partial_t \mathbf{A}_i^{ext} - \nabla_i \left(\Phi_i^{ext} + \Phi_i^{int} \right) + \frac{\mathbf{v}_i}{c} \times (\nabla_i \times \mathbf{A}_i^{ext}) \right] \Big|_{q=q(t)}. \end{aligned} \quad (5.19)$$

and the stochastic accelerations

$$\begin{aligned} m_i \mathbf{a}_i(q(t), t) &= \frac{m_i}{2} [D_* D + D D_*] \mathbf{q}_i(t) \\ &= \left[-\frac{e_i}{c} \partial_t \mathbf{A}_i^{ext} - e_i \nabla_i \left(\Phi_i^{ext} + \Phi_i^{int} \right) \right. \\ &\quad \left. + \frac{e_i}{c} \mathbf{v}_i \times (\nabla_i \times \mathbf{A}_i^{ext}) \right] \Big|_{q=q(t)}, \end{aligned} \quad (5.20)$$

for $i = 1, \dots, N$. Applying the mean derivatives in (5.20), using that $\mathbf{b}_i = \mathbf{v}_i + \mathbf{u}_i$ and $\mathbf{b}_{i*} = \mathbf{v}_i - \mathbf{u}_i$, and replacing $q(t)$ with q in the functions on both sides, straightforward manipulations show that (5.19) turns into

$$\begin{aligned} & \sum_{i=1}^N m_i \left[\partial_t \mathbf{v}_i + \mathbf{v}_i \cdot \nabla_i \mathbf{v}_i - \mathbf{u}_i \cdot \nabla_i \mathbf{u}_i - \frac{\hbar}{2m_i} \nabla_i^2 \mathbf{u}_i \right] \\ &= \sum_{i=1}^N \left[-\frac{e_i}{c} \partial_t \mathbf{A}_i^{ext} - e_i \nabla_i \left(\Phi_i^{ext} + \Phi_i^{int} \right) + \frac{e_i}{c} \mathbf{v}_i \times (\nabla_i \times \mathbf{A}_i^{ext}) \right]. \end{aligned} \quad (5.21)$$

Using (5.10) and (5.12), integrating both sides of (5.21), and setting the arbitrary integration constants equal to zero, we then obtain the N -particle quantum Hamilton–Jacobi equation

$$\begin{aligned} -\partial_t S(q, t) &= \sum_{i=1}^N \frac{[\nabla_i S(q, t) - \frac{e_i}{c} \mathbf{A}_i^{ext}]^2}{2m_i} \\ &\quad + \sum_{i=1}^N e_i \left[\Phi_i^{ext} + \Phi_i^{int} \right] - \sum_{i=1}^N \frac{\hbar^2}{2m_i} \frac{\nabla_i^2 \sqrt{\rho(q, t)}}{\sqrt{\rho(q, t)}}, \end{aligned} \quad (5.22)$$

which describes the total energy of the possible mean trajectories of the particles, and, upon evaluation at $q = q(t)$, the total energy of the actual particles along

their mean trajectories. So (5.11) and (5.22) together define the N -particle HJM equations.

Let us now combine (5.11) and (5.22) into an N -particle Schrödinger equation and write down the most general form of the N -particle wave function. To do this, we first need to impose the N -particle quantisation condition

$$\sum_{i=1}^N \oint_L \nabla_i S(q, t) \cdot d\mathbf{q}_i = nh. \quad (5.23)$$

(For a detailed justification of condition (5.23), cf. Derakhshani (2017).) Then we can combine (5.11) and (5.22) into

$$i\hbar \frac{\partial \psi(q, t)}{\partial t} = \sum_{i=1}^N \left[\frac{[-i\hbar \nabla_i - \frac{e_i}{c} \mathbf{A}_i^{ext}]^2}{2m_i} + e_i (\Phi_i^{ext} + \Phi_i^{int}) \right] \psi(q, t), \quad (5.24)$$

where $\psi = \sqrt{\rho} e^{iS/\hbar} = e^{\frac{1}{\hbar}(R+iS)}$. Our resolution to the problem of multi-time correlations will depend on being able to calculate Nelson trajectories from a quantum-mechanical wave function, so it is essential for us that (5.23) is imposed.

5.3 Nelson's Argument

Here we describe Nelson's argument for the case of a one-dimensional harmonic oscillator, as discussed by Blanchard et al. (1986). Although Nelson (1986) considered the case of two non-interacting one-dimensional harmonic oscillators with the same frequency, the one-dimensional oscillator case illustrates the same point more simply.⁶

Suppose that the one-dimensional oscillator is in the ground state with wave function

$$\psi(x, t) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{\left[-\frac{1}{2}\left(i\omega_0 t + \frac{x^2}{2\sigma^2}\right)\right]}, \quad (5.25)$$

⁶ Nelson's example is interesting because it brings in considerations of nonlocality. We return to these in our concluding Sect. 5.6.

where $\sigma := \hbar/2m\omega_0$. For the oscillator in the ground state, the forward stochastic differential equation is (Nelson 1967)

$$\begin{aligned} dX(t) &= \left(\frac{1}{m} \frac{dS(x, t)}{dx} + \frac{1}{m} \frac{dR(x, t)}{dx} \right) \Big|_{x=X(t)} dt + dW(t) \\ &= -\omega_0 X(t) dt + dW(t), \end{aligned} \quad (5.26)$$

where $S = \frac{\hbar}{2i} \ln(\frac{\psi}{\psi^*})$ and $R = \hbar \ln |\psi|$. Solving (5.26) by quadrature yields

$$X(t) = e^{-\omega_0 t} \left[X(0) + \int_0^t e^{\omega_0 t'} dW(t') \right], \quad (5.27)$$

where we assume $\omega_0 > 0$ and $m > 0$. The two-time correlation function obtained using (5.27) is thus

$$E[X(0)X(t)] = \sigma^2 e^{-\omega_0 t} \quad (5.28)$$

for $t \geq 0$.

Now we calculate the two-time correlation from the Heisenberg picture of textbook quantum mechanics. For the oscillator Hamiltonian

$$\hat{H} = -\frac{\hbar}{2m} \partial_x^2 + \frac{1}{2} m \omega_0^2 \hat{x}^2 \quad (5.29)$$

the Heisenberg position operator at time t is defined as

$$\begin{aligned} \hat{x}(t) &= e^{\frac{i}{\hbar} \hat{H} t} \hat{x}(0) e^{-\frac{i}{\hbar} \hat{H} t} \\ &= \hat{x}(0) \cos(\omega_0 t) + \frac{\sin(\omega_0 t)}{m\omega_0} (-i\hbar \partial_x). \end{aligned} \quad (5.30)$$

Hence the commutator

$$[\hat{x}(0), \hat{x}(t)] = i\hbar \frac{\sin(\omega_0 t)}{m\omega_0}, \quad (5.31)$$

which vanishes for $t = n\pi/\omega_0$, where $n \in \mathbb{Z}$. For these times, and assuming the ground state wave function (5.25), the two-time correlation function becomes

$$\langle \psi | \hat{x}(0)\hat{x}(t) | \psi \rangle = (-1)^n \sigma^2, \quad (5.32)$$

which clearly differs from (5.28).⁷ In particular, the latter vanishes as $t \rightarrow \infty$.

In Nelson's example, since the two harmonic oscillators are non-interacting their position operators (at the same or different times) commute and the above analysis carries over straightforwardly, i.e., one obtains the same disagreement in results. Another worked example, involving a particle with an initially Gaussian wave function scattering through a potential, is given by Blanchard et al. (1986).

5.4 Blanchard et al.'s Resolution

The Nelsonian expression (5.27) has a perfectly coherent meaning as the expectation value of the product of the position of the oscillator at $X(0)$ and $X(t)$ when the oscillator's position is *not* measured, and for this case the two-time correlation is indeed the one with exponential decay, (5.28). On the other hand, as Blanchard et al. (1986) note for Nelson's theory, the collapse of a subsystem's wave function in a position measurement (in particular the first measurement at $t = 0$) changes the mean forward drift velocity in (5.26) or (5.1), which in turn changes the particle's stochastic trajectory. The reason is that the i th drift velocity $\mathbf{b}_i(q(t), t)$ depends on a wave function $\psi = e^{\frac{1}{\hbar}(R+iS)}$ satisfying the Schrödinger equation (5.25) via

$$\mathbf{b}_i(q(t), t) = \frac{\hbar}{m_i} (\text{Re} + \text{Im}) \nabla_i \ln \psi(q, t)|_{q=q(t)}, \quad (5.33)$$

hence the drift velocity after (say) an ideal position measurement of a Nelsonian particle will differ from before the measurement, assuming that the wave function collapses as a consequence of the measurement. With this observation in hand, it is not at all surprising that (5.27) for the unmeasured oscillator's position should differ from the quantum mechanical correlation (5.31), the latter referring to measurements of the oscillator's position.

In the case of the oscillator, Blanchard et al. take its collapsed wave function at $t = 0$, when the ideal position measurement is made with result X_0 , to be a delta function

$$\lim_{t \rightarrow 0^+} \psi^{X_0}(x, t) = \delta(x - X_0) \quad (5.34)$$

⁷ Blanchard et al. (1986) give the general expression for the two-time correlation function when the operators do not commute, see their equation (34). Our Eq. (5.32) is valid for the special times when the operators commute.

which thereafter evolves via the oscillator's Hamiltonian (5.29) to some excited state at finite time $t > 0$. They note that since Schrödinger's equation with the Hamiltonian (5.29) has the propagator (Sakurai and Napolitano 1993, pp. 116–119)

$$K(x, t; x'; 0) = \sqrt{\frac{m\omega_0}{2\pi i\hbar \sin(\omega_0 t)}} \times \exp \left[\left(\frac{im\omega_0}{2\hbar} \right) \times \frac{(x^2 + x'^2) \cos(\omega_0 t) - 2xx'}{\sin(\omega_0 t)} \right], \quad (5.35)$$

with the property that $\lim_{t \rightarrow 0^+} K(x, t; x'; 0) = \delta(x - x')$, the time-evolved wave function for the oscillator at $t > 0$, with initial condition (5.34), is just

$$\psi^{X_0}(x, t) := \int_{-\infty}^{\infty} K(x, t; x'; 0) \psi^{X_0}(x', 0) dx' = K(x, t; X_0, 0). \quad (5.36)$$

(Our form for the propagator in (5.35) differs from the form given in Blanchard et al. (1986); however it can be readily shown, using the definition of the propagator (Sakurai and Napolitano 1993, pp. 116–119), that the two forms are equivalent. Our form for the propagator makes the calculation of the drift velocity more transparent.) Using (5.36) in (5.33), we can calculate the drift velocity for the oscillator at time t and obtain

$$b^{X_0}(x, t) = \omega_0 \left[\frac{x}{\tan(\omega_0 t)} - \frac{X_0}{\sin(\omega_0 t)} \right]. \quad (5.37)$$

Inserting this drift velocity in the stochastic differential equation

$$dX(t) = b^{X_0}(X(t), t) dt + dW(t) \quad (5.38)$$

and solving by quadrature results in a periodic oscillator trajectory for $t > 0$:

$$X(t) = [\cos(\omega_0 t) - \sin(\omega_0 t) \cot(\omega_0 s)] X_0 + \frac{\sin(\omega_0 t)}{\sin(\omega_0 s)} X(s) + \sin(\omega_0 t) \int_s^t \frac{dW(t')}{\sin(\omega_0 t')}, \quad (5.39)$$

for $0 < s \leq t$. Clearly, for $t = n\pi/\omega_0$, the trajectory $X(t)$ is just given by $(-1)^n X_0$. So the stochastic mechanical two-time correlation $E[X(0)X(t)]$, for $t = n\pi/\omega_0$ and using again the probability density $|\psi(x, t)|^2$ to compute the

absolute expectation, where $\psi(x, t)$ is the ground-state wave function (5.25) prior to the position measurement, reads⁸

$$E[X(0)X(t)] = (-1)^n \sigma^2, \quad (5.40)$$

which agrees with the Heisenberg picture result (5.32).

We note that the drift velocity (5.34) for a wave function corresponding to a delta function is undefined. Of course, the assumption that the wave function collapses to a delta function in an ideal position measurement is itself an idealisation. A more realistic assumption is that the wave function collapses to a Gaussian of some narrow width w . For such a function, the drift velocity exists and is well-behaved. If we make this more realistic assumption, i.e., that $\lim_{t \rightarrow 0^+} \psi^{X_0}(x, t) = g(x - \tilde{x})$, where $g(x - \tilde{x})$ is a normalised Gaussian function of width w centered on some point $\tilde{x}$ at $t = 0$, we need only multiply the propagator $K(x, t; x'; 0)$ with an initial wave function corresponding to $g(x' - \tilde{x})$ and integrate over all space (that, is over x') to obtain the oscillator's wave function at $t > 0$. It is easy to confirm that, for such a wave function, the resulting drift velocity and oscillator trajectory for $t > 0$ will be periodic. It is therefore reasonable to expect that the quantum mechanical two-time correlation will also be reproduced.

The obvious question is whether one can justify using collapsed wave functions in stochastic mechanics. At first sight this seems incoherent: the wave function in stochastic mechanics is a derived notion, and one may not introduce separate postulates for its evolution. As we have seen, the evolution of the wave function in stochastic mechanics is given by the Schrödinger equation (5.24).

On the other hand, the stochastic mechanics calculation in the previous section does not model measurements using the Schrödinger equation. Indeed, the calculation assumes that the oscillator is following the free Schrödinger evolution throughout. Measurement is apparently taken into account by conditioning on $X(0)$, but that means one is assuming that a measurement simply reveals the oscillator's position without affecting in any way its subsequent dynamics. This assumption is surely not licensed. One needs to model the measurement explicitly and check whether the resulting drift equals that given by the collapsed wave function of textbook quantum mechanics.

Blanchard et al. (1986) make one possible suggestion in this regard. Precisely since the wave function in stochastic mechanics is a derived notion, if as a consequence of the measurement one updates the distribution ρ , this will lead to a different wave function also in stochastic mechanics. Such 'non-equilibrium' constraints on the distribution of the process can indeed be modelled using a new wave function, but require introducing also a non-linearity in the resulting Schrödinger equation (Bacciagaluppi 2012).

⁸ We note that this expression also agrees with the more general expression for the two-time correlation given by Blanchard et al. (1986), see equation (33) therein.

In the next section, we shall provide a justification for the use of effectively collapsed wave functions fashioned after that used in de Broglie and Bohm's pilot-wave theory, explicitly modelling measurements by inclusion of appropriate degrees of freedom of the measuring apparatus.

5.5 Effective Collapse Resolution

How does pilot-wave theory treat measurements and recover the appearance of collapse, even though there is no 'true' collapse in the theory?

We shall start by treating a very simple case, that of performing a position measurement by letting a particle go through a slit. Assume the particle, initially guided by a wave $\psi(t)$, goes through the slit at time $t = 0$. At times larger than 0, the pilot wave has developed one 'transmitted' and one 'reflected' component:

$$\psi(t) = \psi_T(t) + \psi_R(t). \quad (5.41)$$

It is important to note that these two components do not overlap. As a consequence, if the particle indeed goes through at $t = 0$, for times $t > 0$ the only component of the pilot wave that is relevant for the motion of the particle is the transmitted component. By the same token, if the particle fails to go through, the only component of the pilot wave that is relevant for the future motion of the particle is the reflected component. For all effects and purposes (which in pilot-wave theory means: for the purpose of guiding the motion of the particle), the passage or failed passage through the slit has collapsed the wave.

If we translate this to the framework of stochastic mechanics, we see that through the simple inclusion of the external potential, conditioning on $\mathbf{x}(0)$ in effect picks out a collapsed wave function. If we compare the predictions of stochastic mechanics with those of quantum mechanics, the predictions of the two theories are identical. But in this simple case it is also unsurprising, because the quantum mechanical predictions for later position measurements are insensitive to whether or not we actually detect the particle at the slit and collapse the wave function. Thus, it is actually immaterial whether stochastic mechanics in this case is able to recover the notion of collapse.

This simple model, simply using an external potential, is essentially how de Broglie (1928) could treat diffraction phenomena. But it fails already for slightly more complex measurements.

Imagine a double-slit experiment. Simply including the external potential covers the case in which both slits are open. But now it makes all the difference whether or not we detect *which* open slit the particle goes through. In order to model this in pilot-wave theory (and by extension in stochastic mechanics), we need to include also *apparatus degrees of freedom*.

Let us do so, and let the particle interact with an 'apparatus' with initial wave function $\phi(\mathbf{y}, t)$ for measuring the particle's position to within the aperture of the

U(pper) or L(ower) slit. We schematically model the interaction as:

$$\psi(\mathbf{x}, t)\phi(\mathbf{y}, t) \rightarrow a \psi_U(\mathbf{x}, t)\phi_U(\mathbf{y}, t) + b \psi_L(\mathbf{x}, t)\phi_L(\mathbf{y}, t). \quad (5.42)$$

Note again that the two components of the wave do not overlap. So, analogously as above, if the particle goes through the upper or lower slit its motion is guided, respectively, by the effectively collapsed wave functions $\psi_U(\mathbf{x}, t)$ or $\psi_L(\mathbf{x}, t)$. This time, however, this is crucially due to the explicit inclusion of the apparatus degrees of freedom: $\psi_U(\mathbf{x}, t)$ and $\psi_L(\mathbf{x}, t)$ will spread out from the slits and overlap, but the macroscopically different $\phi_U(\mathbf{y}, t)$ and $\phi_L(\mathbf{y}, t)$ do not (or negligibly so), thus ensuring the non-overlap of the two components in (5.42).

Again, translating to the case of stochastic mechanics, conditioning on $\mathbf{x}(0)$ effectively picks out the collapsed wave function $\psi_U(\mathbf{x}, t)$ or $\psi_L(\mathbf{x}, t)$, and the predictions of stochastic mechanics coincide with those of standard quantum mechanics with collapse. This example generalises to arbitrary sequences of position measurements, thereby solving Nelson's problem of multi-time correlations: whenever position is actually measured in such a way that the measurement result is encoded in the degrees of freedom of some macroscopic object, the multi-time correlations calculated in stochastic mechanics will coincide with those calculated in standard quantum mechanics.

For completeness, let us consider also measurements of observables other than position. Take for instance a measurement not of the position but of the component of the particle's momentum parallel to the slits (maybe using a suspended double slit). This might be modelled as:

$$\psi(\mathbf{x}, t)\phi(\mathbf{y}, t) \rightarrow c \psi_\uparrow(\mathbf{x}, t)\phi_\uparrow(\mathbf{y}, t) + d \psi_\downarrow(\mathbf{x}, t)\phi_\downarrow(\mathbf{y}, t). \quad (5.43)$$

As in the case of (5.42), even though of course $\psi_\uparrow(\mathbf{x}, t)$ and $\psi_\downarrow(\mathbf{x}, t)$ overlap, the two components of (5.43) do not overlap because $\phi_\uparrow(\mathbf{y}, t)$ and $\phi_\downarrow(\mathbf{y}, t)$ do not. Therefore, after the measurement the particle will be guided by either $\psi_\uparrow(\mathbf{x}, t)$ or $\psi_\downarrow(\mathbf{x}, t)$, depending on the result of the measurement as encoded in $\phi_\uparrow(\mathbf{y}, t)$ and $\phi_\downarrow(\mathbf{y}, t)$, respectively.

In the case of stochastic mechanics, conditioning on $\mathbf{x}(0)$ no longer picks out a collapsed wave function, but the measurement was not a measurement of position, so $\mathbf{x}(0)$ does not encode the result of the measurement. It is $\mathbf{y}(0)$ that does, and conditioning on $\mathbf{y}(0)$ accordingly picks out the collapsed wave function $\psi_\uparrow(\mathbf{x}, t)$ or $\psi_\downarrow(\mathbf{x}, t)$.

The treatment of measurements and of 'effective collapse' we have just sketched was Bohm's (1952) crucial contribution to pilot-wave theory. It allows the theory to recover the predictions of quantum mechanics in all cases of measurement, as long as the measurement results are encoded in some macroscopic positions (whether the positions of pointers or of ink drops on a printout).

In stochastic mechanics, it explains how the drift velocity $\mathbf{b}$ after a measurement is the one corresponding to the usual collapsed wave function of standard quantum mechanics. The problem of multi-time correlations disappears.

5.6 Conclusion

Of the two traditional problems besetting stochastic mechanics, Wallstrom's (1994) of the apparent ad hoc character of the quantisation of the wave function and Nelson's (1986) of multi-time correlations, Nelson himself considered the latter to be the most pressing.⁹ We believe both of these problems have been addressed in a satisfactory way. In particular we have pointed out in this paper how the explicit inclusion of measurements in stochastic mechanics resolves the problem of multi-time correlations. Nevertheless, we presume Nelson would not have been entirely happy, because the resolution of this problem comes at the price of nonlocality.

In both Nelson (1986) and Nelson (2005) the problem of multi-time correlations is presented as a Bell-type dilemma between reproducing the quantum mechanical predictions and locality. Instead of taking a single one-dimensional oscillator, as we did in Sect. 5.3, Nelson takes two non-interacting one-dimensional oscillators of the same frequency ω_0 that are entangled in such a way that at $t = 0$ their positions are, say, perfectly correlated. The two oscillators may, however, be at arbitrary separation along some other dimension.

Since $\hat{x}_2(t)$ is periodic, so is the quantum mechanical correlation function $\langle \psi_0 | \hat{x}_1(0) \hat{x}_2(t) | \psi_0 \rangle$. But, as with (5.28), the Nelsonian correlation function $E[X_1(0)X_2(t)]$ will have an exponential decay in time. Thus, in general,

$$E[X_1(0)X_2(t)] \neq \langle \psi_0 | \hat{x}_1(0) \hat{x}_2(t) | \psi_0 \rangle. \quad (5.44)$$

And the only way of getting equality is if the measurement on the first oscillator at $t = 0$ somehow nonlocally affects the later trajectory of the second oscillator in some appropriate way.

As we have seen, including the measurement apparatus in the stochastic mechanical description automatically yields the same predictions as the standard textbook collapse. In the case of two entangled systems, it will reproduce the collapse at a distance (as is well-known from pilot-wave theory). Thus, a measurement on one oscillator does have an effect on the drift velocity of the other oscillator. We note, incidentally, that the same mechanism is what enables stochastic mechanics to violate the Bell inequalities.¹⁰ In conclusion, stochastic mechanics is nonlocal and the dilemma is resolved in the way Nelson would have liked least.

⁹ Personal communication to one of us (GB) at a conference in Vienna in November 2011.

¹⁰ This also addresses the objection to stochastic mechanics raised by Kiukas and Werner (2010), who discuss violations of the Bell inequalities based on measurements of position at different times and suggest that the existence of joint multi-time distributions in stochastic mechanics would prevent it from violating these Bell inequalities.

Appendix: Computation of Forward and Backward Stochastic Derivatives

To compute $D\mathbf{b}_i(q(t), t)$ (or $D_*\mathbf{b}_i(q(t), t)$) expand $\mathbf{b}_i$ in a Taylor series up to terms of order two in $d\mathbf{q}_i(t)$:

$$\begin{aligned} d\mathbf{b}_i(q(t), t) &= \frac{\partial \mathbf{b}_i(q(t), t)}{\partial t} dt + \sum_{i=1}^N d\mathbf{q}_i(t) \cdot \nabla_i \mathbf{b}_i(q, t)|_{q=q(t)} \\ &+ \sum_{i=1}^N \frac{1}{2} \sum_{n,m} dq_{in}(t) dq_{im}(t) \frac{\partial^2 \mathbf{b}_i(q, t)}{\partial q_{in} \partial q_{im}}|_{q=q(t)} + \dots \end{aligned} \quad (5.45)$$

From (5.1), we can replace $d\mathbf{q}_i(t)$ by $d\mathbf{W}_i(t)$ in the last term, and when taking the conditional expectation at time t in (5.13), we can replace $d\mathbf{q}_i(t) \cdot \nabla_i \mathbf{b}_i|_{q=q(t)}$ by $\mathbf{b}_i(\mathbf{q}(t), t) \cdot \nabla_i \mathbf{b}_i|_{q=q(t)}$ since $d\mathbf{W}_i(t)$ is independent of $\mathbf{q}_i(t)$ and has mean 0. From (5.2), we then obtain

$$D\mathbf{b}_i(q(t), t) = \left[\frac{\partial}{\partial t} + \sum_{i=1}^N \mathbf{b}_i(q(t), t) \cdot \nabla_i + \sum_{i=1}^N \frac{\hbar}{2m_i} \nabla_i^2 \right] \mathbf{b}_i(q(t), t), \quad (5.46)$$

and likewise

$$D_*\mathbf{b}_{i*}(q(t), t) = \left[\frac{\partial}{\partial t} + \sum_{i=1}^N \mathbf{b}_{i*}(q(t), t) \cdot \nabla_i - \sum_{i=1}^N \frac{\hbar}{2m_i} \nabla_i^2 \right] \mathbf{b}_{i*}(q(t), t). \quad (5.47)$$

References

- Bacciagaluppi, G. 2003. Derivation of the symmetry postulates for identical particles from pilot-wave theories. <https://arxiv.org/pdf/quant-ph/0302099.pdf>
- Bacciagaluppi, G. 2005. *Endophysics, Time, Quantum and the Subjective*, 367–388. Singapore: World Scientific. Revised version: <http://philsci-archive.pitt.edu/8853/>
- Bacciagaluppi, G. 2012. *Journal of Physics: Conference Series* 361: 012017, 1–12.
- Blanchard, P., S. Golin, and M. Serva. 1986. *Physical Review D* 34 (12): 3732–3738.
- Blanchard, P., M. Cini, and M. Serva. 1991. *BiBoS-463/91*, 149–171 Bielefeld: BiBoS.
- Bohm, D. 1952. *Physical Review* 85 (2): 166–179 and 180–193.
- de Broglie, L. 1928. *Electrons et photons*, 105–141. Paris: Gauthier-Villars.
- de la Peña, L., and A.M. Cetto. 1982. *Foundations of Physics* 12: 1017–1037.
- Derakhshani, M. 2017. PhD Thesis (Utrecht). <https://arxiv.org/abs/1804.01394>
- Goldstein, S. 1987. *Journal of Statistical Physics* 47 (5/6): 645–667.
- Kiukas, J., and R.F. Werner. 2010. *Journal of Mathematical Physics* 51 (7): 072105, 1–16.
- Nelson, E. 1966. *Physical Review* 150 (4): 1079–1085.
- Nelson, E. 1967. *Dynamical Theories of Brownian Motion*. Princeton: PUP.

- Nelson, E. 1985. *Quantum Fluctuations*. Princeton: PUP.
- Nelson, E. 1986. Field theory and the future of stochastic mechanics. https://doi.org/10.1007/3540171665_87
- Nelson, E. 2005. The mystery of stochastic mechanics. <https://web.math.princeton.edu/~nelson/papers/talk.pdf>
- Peruzzi, G., and A. Rimini. 1996. *Foundations of Physics Letters* 9 (6): 505–519.
- Sakurai, J.J., and J. Napolitano. 1993. *Modern Quantum Mechanics*. Revised ed. Boston: Addison-Wesley.
- Schmelzer, I. 2011. An answer to the Wallstrom objection against Nelsonian stochastics. <https://arxiv.org/abs/1101.5774v3>
- Wallstrom, T. 1994. *Physical Review A* 49 (3): 1613–1617.
- Yasue, K. 1981. *Journal of Mathematical Physics* 22 (5): 1010–1020.