Uniform Interpolation via Nested Sequents

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Abstract. A modular proof-theoretic framework was recently developed to prove Craig interpolation for normal modal logics based on generalizations of sequent calculi (e.g., nested sequents, hypersequents, and labelled sequents). In this paper, we turn to uniform interpolation, which is stronger than Craig interpolation. We develop a constructive method for proving uniform interpolation via nested sequents and apply it to reprove the uniform interpolation property for normal modal logics K, D, and T. While our method is proof-theoretic, the definition of uniform interpolation for nested sequents also uses semantic notions, including bisimulation modulo an atomic proposition.

Keywords: Uniform interpolation · Modal logic · Nested sequents

1 Introduction

A propositional (modal) logic L admits the Craig interpolation property (CIP) if for any formulas φ and ψ such that $\vdash_{\mathsf{L}} \varphi \to \psi$, there is an interpolant θ containing only atomic propositions that occur in both φ and ψ such that $\vdash_{\mathsf{L}} \varphi \to \theta$ and $\vdash_{\mathsf{L}} \theta \to \psi$. Logic L has the uniform interpolation property (UIP) if for each formula φ and each atomic proposition p there are uniform interpolants $\exists p\varphi$ and $\forall p\varphi$ built from atomic propositions occurring in φ except for p, such that for all formulas ψ not containing p:

$$\vdash_{\mathsf{L}} \varphi \to \psi \quad \Leftrightarrow \vdash_{\mathsf{L}} \exists p\varphi \to \psi \qquad \text{and} \qquad \vdash_{\mathsf{L}} \psi \to \varphi \quad \Leftrightarrow \vdash_{\mathsf{L}} \psi \to \forall p\varphi.$$

It is well known that this property is stronger than Craig interpolation.

To prove the CIP (UIP) constructively, one can use analytic (terminating) sequent calculi. Whereas for the CIP the syntactic proofs are often straightforward, the case of the UIP is more complicated. Pitts provided the first syntactic proof of this kind, establishing the UIP for IPC [19]. Bílková successfully adjusted the method to (re)prove the UIP for several modal logics including K, T, and GL [2]. Iemhoff provided a modular method for (intuitionistic)

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modal logics and intermediate logics based on sequent calculi consisting of the so-called focused rules, among others establishing the UIP for D [12,13].

There are also algebraic and model-theoretic methods. The UIP for GL and K is due to Shavrukov [21] and Ghilardi [8] respectively. Interestingly, modal logics S4 and K4 do not enjoy the UIP [2,9] despite enjoying the CIP. Visser provided semantic proofs for K, GL, and IPC based on bounded bisimulation up to atomic propositions [24]. This method was later used for the stronger Lyndon UIP [14]. The semantic interpretation of uniform interpolation is called bisimulation quantifiers, see [6] for an overview. Bisimulation will also play a role in this paper.

The proof-theoretic approach has two advantages. First, it enables one to find interpolants constructively rather than merely prove their existence.[1] Second, negative results were obtained in [12,13] stating that logics without the UIP cannot have certain natural sequent calculi. As a consequence, K4 and S4 do not possess such proof systems. Similar negative results were obtained for modal and substructural logics in [22] and [23] using the CIP and UIP.

The goal of this paper is to extend the same line of research to multisequent formalisms starting with nested sequents[2]. Multisequent formalisms, such as nested sequents, hypersequents, and labelled sequents, are (commonly believed to be) more expressive than sequents and offer modular and analytic calculi for a wide range of logics. E.g., S5 has well-known cut-free hypersequent calculi [1,17] but no known cut-free sequent calculus while modal logics K5 and B possess cut-free nested sequent calculi, but no hypersequent calculi [5]. Nested sequent calculi were recently used to prove the CIP for modal logics [7]. A modular proof-theoretic framework encompassing them and other multisequents was provided in [15]. The same ideas, which combine syntactic and semantic reasoning, were extended to multisequent calculi for intermediate logics [16].

We provide a method to prove the UIP for K, D, and T using terminating nested sequent calculi from [5]. These calculi are used to construct uniform interpolants syntactically, whereas the correctness proof for the constructed interpolants relies on semantic reasoning, including model modifications and bisimulation. While the UIP for these three logics has been previously shown via analytic sequent calculi, our constructive method is also applicable to logics based on multisequent formalisms that lack a sequent representation. In particular, in an extended version [10] of this paper, we successfully adapted our method to hypersequents for S5.

Bílková [3] also provided a syntactic proof for the UIP for K based on nested sequents. The main difference with our method is that we exploit the treelike structure of nested sequents, thus reflecting the treelike models for K, by using semantic reasoning while the algorithm for the interpolants remains fully syntactic.

The paper is organized as follows. In Sect. 2 the nested sequent calculi for K, T, and D, as well as model modifications invariant under bisimulation, are

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[1] More precisely, it enables one to find interpolants efficiently rather than by an exhaustive search that terminates due to the existence of the interpolant.

[2] Nested sequents are also known as tree-hypersequents [20] or deep sequents [5].

introduced. In Sect. 3, we prove uniform interpolation for K, T, and D. Finally, in Sect. 4 we summarize the results and outline future work. An extended version [10] of this paper provides more detailed proofs of these results and, in addition, includes a direct proof of the UIP for S5 via hypersequents.

2 Preliminaries

Definition 1. Modal formulas *in* negation normal form *are defined by the grammar* $\varphi ::= \bot \mid \top \mid p \mid \overline{p} \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Box\varphi \mid \Diamond\varphi$ *where* $\bot$ *and* $\top$ *are* Boolean constants*,* p *is an* atomic proposition (atom) *from a countable set* Prop*, and* $\overline{p}$ *is its negation. An element* ℓ *of the set* Lit *of* literals *is either an atom or its negation. Literals and Boolean constants are* atomic formulas*.*

We define $\overline{\varphi}$ (or $\neg\varphi$) recursively as usual using De Morgan's laws to push the negation inwards. We define $\varphi \to \psi := \overline{\varphi} \vee \psi$ as usual.

Definition 2. *A* nested sequent Γ *is recursively defined in the following form:*

$$\varphi_1, \ldots, \varphi_n, [\Gamma_1], \ldots, [\Gamma_m]$$

where $\varphi_1, \ldots, \varphi_n$ *are modal formulas for* $n \geq 0$ *and* $\Gamma_1, \ldots, \Gamma_m$ *are nested sequents for* $m \geq 0$*. We call brackets* [ ] *a* structural box*. The* formula interpretation ι *of a nested sequent is defined recursively by*

$$\iota(\varphi_1, \ldots, \varphi_n, [\Gamma_1], \ldots, [\Gamma_m]) := \varphi_1 \vee \cdots \vee \varphi_n \vee \Box\iota(\Gamma_1) \vee \cdots \vee \Box\iota(\Gamma_m).$$

One way of looking at a nested sequent is to consider a tree of ordinary (one-sided) sequents, i.e., multisets of formulas. Each structural box in the nested sequent creates a *child* in the tree. In order to be able to reason about formulas in a particular tree node, we introduce labels. A *label* is a finite sequence of natural numbers. We denote labels by $\sigma, \tau, \ldots$; a label $\sigma * n$ denotes the label σ extended by the natural number n. We sometimes write σn instead of $\sigma * n$, unless it is ambiguous, as, e.g., for $1 * 2 * 3$, which is different from $1 * 23$.

Definition 3 (Labeling). *For a nested sequent* Γ *and label* σ *we define a* labeling function l_σ *to recursively label structural boxes in nested sequents as follows:*

$$l_\sigma(\varphi_1, \ldots, \varphi_n, [\Gamma_1], \ldots, [\Gamma_m]) := \varphi_1, \ldots, \varphi_n, [l_{\sigma 1}(\Gamma_1)]_{\sigma 1}, \ldots, [l_{\sigma m}(\Gamma_m)]_{\sigma m}.$$

Let $\mathcal{L}_\sigma(\Gamma)$ *be the set of labels occurring in* $l_\sigma(\Gamma)$ *plus label* σ *(for formulas outside all structural boxes). Define* $l(\Gamma) := l_1(\Gamma)$*, and let* $\mathcal{L}(\Gamma) := \mathcal{L}_1(\Gamma)$*.*[3]

Formulas in a nested sequent Γ *are* labeled *according to the labeling of the structural boxes containing them. We write* $1 : \varphi \in \Gamma$ *iff the formula* φ *occurs in* Γ *outside all structural boxes. Otherwise,* $\sigma : \varphi \in \Gamma$ *whenever* φ *occurs in* $l(\Gamma)$ *within a structural box labeled* σ*.*

[3] Labeled nested sequents are closely related to labelled sequents from [18] but retain the nested notation.

$$\mathsf{id_P}\ \frac{}{\Gamma\{p,\overline{p}\}} \qquad \mathsf{id}_\top\ \frac{}{\Gamma\{\top\}} \qquad \vee\ \frac{\Gamma\{\varphi\vee\psi,\varphi,\psi\}}{\Gamma\{\varphi\vee\psi\}} \qquad \wedge\ \frac{\Gamma\{\varphi\wedge\psi,\varphi\}\quad \Gamma\{\varphi\wedge\psi,\psi\}}{\Gamma\{\varphi\wedge\psi\}}$$

$$\Box\ \frac{\Gamma\{\Box\varphi,[\varphi]\}}{\Gamma\{\Box\varphi\}} \qquad \mathsf{k}\ \frac{\Gamma\{\Diamond\varphi,[\Delta,\varphi]\}}{\Gamma\{\Diamond\varphi,[\Delta]\}} \qquad \mathsf{d}\ \frac{\Gamma\{\Diamond\varphi,[\varphi]\}}{\Gamma\{\Diamond\varphi\}} \qquad \mathsf{t}\ \frac{\Gamma\{\Diamond\varphi,\varphi\}}{\Gamma\{\Diamond\varphi\}}$$

Fig. 1. Terminating nested rules: the principal formula is not K- (D-, T-) saturated.

The set $\mathcal{L}(\Gamma)$ can be seen as the set of nodes of the corresponding tree of Γ, with 1 being the root. Often, we do not distinguish between a nested sequent Γ and its labeled sequent $l(\Gamma)$. For instance, we write $\sigma \in \Gamma$ if $\sigma \in \mathcal{L}(\Gamma)$.

Whether X is a formula, a sequence/set/multiset of formulas, a nested sequent/context, or some other formula-based object, we denote by $Var(X) \subseteq$ Prop the set of atoms occurring in X (note that p may also occur in the form of $\overline{p}$).

Recall that the normal modal logic K consists of all classical tautologies, the k-axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and is closed under *modus ponens* (from $\varphi \to \psi$ and φ, infer ψ) and *necessitation* (from φ, infer $\Box\varphi$). Further, the modal logics D and T are defined as $\mathsf{D} := \mathsf{K} + \Box\varphi \to \Diamond\varphi$ and $\mathsf{T} := \mathsf{K} + \Box\varphi \to \varphi$.

The nested calculus NK for the modal logic K consists of the rules in the first row in Fig. 1 plus the rules $\Box$ and k. This calculus is an extension of the multiset-based version from [5] to the language with Boolean constants $\bot$ and $\top$, necessitating the addition of the rule $\mathsf{id}_\top$ for handling these. The calculus ND (NT) for the logic D (T) is obtained by adding to NK the rule d (t). As shown in [5], the nested sequent calculi NK, ND, and NT are sound and complete for modal logics K, D, and T respectively.

Definition 4 (Saturation). *Let $\Gamma = \Gamma'\{\theta\}_\sigma$, i.e., $\sigma : \theta \in \Gamma$. The formula θ is* K-saturated in *Γ if the following conditions hold based on the form of θ:*

- *θ is an atomic formula;*
- *if $\theta = \varphi \vee \psi$, then both $\sigma : \varphi \in \Gamma$ and $\sigma : \psi \in \Gamma$;*
- *if $\theta = \varphi \wedge \psi$, then either $\sigma : \varphi \in \Gamma$ or $\sigma : \psi \in \Gamma$;*
- *if $\theta = \Box\varphi$, then there is a label $\sigma n \in \mathcal{L}(\Gamma)$ such that $\sigma n : \varphi \in \Gamma$.*

The formula $\theta = \Diamond\varphi$ is

- K-saturated in Γ w.r.t. $\sigma n \in \mathcal{L}(\Gamma)$ *if $\sigma n : \varphi \in \Gamma$;*
- D-saturated in Γ *if there is some label $\sigma n \in \mathcal{L}(\Gamma)$;*
- T-saturated in Γ *if $\sigma : \varphi \in \Gamma$.*

A nested sequent Γ is K-saturated *if* (1) *it is neither of the form $\Gamma'\{p, \overline{p}\}$ for some $p \in$ Prop nor of the form $\Gamma'\{\top\}$; and* (2) *all its formulas $\sigma : \Diamond\varphi$ are K-saturated w.r.t. every child of σ; and* (3) *all its other formulas are K-saturated in Γ. A nested sequent is D-saturated (T-saturated) if it is K-saturated and all its formulas $\sigma : \Diamond\varphi$ are D-saturated (T-saturated) in Γ.*

Theorem 5 ([5]). *The calculi* NK*,* ND*, and* NT *in Fig. 1 are terminating.*

Definition 6. *A* Kripke model *is a triple $\mathcal{M} = (W, R, V)$, where $W \neq \varnothing$, $R \subseteq W \times W$, and $V : \mathsf{Prop} \to 2^W$ is a* valuation function*. Define $\mathcal{M}, w \models \varphi$ as usual: $\mathcal{M}, w \models \top$ and $\mathcal{M}, w \not\models \bot$; for $p \in \mathsf{Prop}$, we have $\mathcal{M}, w \models p$ iff $w \in V(p)$ and $\mathcal{M}, w \models \overline{p}$ iff $w \notin V(p)$; we have $\mathcal{M}, w \models \varphi \wedge \psi$ ($\mathcal{M}, w \models \varphi \vee \psi$) iff $\mathcal{M}, w \models \varphi$ and (or) $\mathcal{M}, w \models \psi$; finally, $\mathcal{M}, w \models \Box\varphi$ iff $\mathcal{M}, v \models \varphi$ whenever wRv and $\mathcal{M}, w \models \Diamond\varphi$ iff $\mathcal{M}, v \models \varphi$ for some wRv. A formula φ is* valid in $\mathcal{M}$*, denoted $\mathcal{M} \models \varphi$, when $\mathcal{M}, w \models \varphi$ for all $w \in W$.*

A model $\mathcal{M}' = (W', R', V')$ is a submodel *of $\mathcal{M} = (W, R, V)$ when $W' \subseteq W$, $R' = R \cap (W' \times W')$, and $V'(p) = V(p) \cap W'$ for each $p \in \mathsf{Prop}$. A* submodel generated by *$w \in W$, denoted $\mathcal{M}_w := (W_w, R_w, V_w)$, is the smallest submodel $\mathcal{M}' = (W', R', V')$ of $\mathcal{M}$ such that $w \in W'$ and $v \in W'$ when xRv and $x \in W'$.*

We will use models based on finite intransitive directed trees, usually denoting the *root* ρ. For K, we require the accessibility relation R to be irreflexive, i.e., $\forall w \in W \neg(wRw)$. For T, R is reflexive, i.e., $\forall w \in W wRw$. And for D, R is serial, i.e., $\forall w \in W \exists v \in W wRv$. Note that seriality implies reflexivity of the leaves of the tree. We call these models K-models, T-models, and D-models respectively.

Theorem 7 ([11, Sect. 4.20]). *If* $\mathsf{L} \in \{\mathsf{K}, \mathsf{D}, \mathsf{T}\}$*, then $\varphi \in \mathsf{L}$ iff $\mathcal{M} \models \varphi$ for each* L*-model $\mathcal{M}$.*

Following [15], we extend definitions of truth and validity to nested sequents, recall relevant facts about bisimulation, and introduce some model modifications.

Definition 8. *A* (treelike) multiworld interpretation of a nested sequent Γ into a model $\mathcal{M} = (W, R, V)$ *is a function $\mathcal{I} : \mathcal{L}(\Gamma) \to W$ from labels in Γ to worlds of $\mathcal{M}$ such that $\mathcal{I}(\sigma) R \mathcal{I}(\sigma n)$ whenever $\{\sigma, \sigma n\} \subseteq \mathcal{L}(\Gamma)$. Then*

$$\mathcal{M}, \mathcal{I} \models \Gamma \qquad \Longleftrightarrow \qquad \mathcal{M}, \mathcal{I}(\sigma) \models \varphi \text{ for some } \sigma : \varphi \in \Gamma.$$

Γ is valid in $\mathcal{M}$*, denoted by $\mathcal{M} \models \Gamma$, means that $\mathcal{M}, \mathcal{I} \models \Gamma$ for all multiworld interpretations $\mathcal{I}$ of Γ into $\mathcal{M}$.*

The following lemma, which can be easily proved by induction on the structure of Γ, implies completeness for validity of nested sequents.

Lemma 9. *$\mathcal{M} \models \Gamma$ iff $\mathcal{M} \models \iota(\Gamma)$ for any nested sequent Γ and model $\mathcal{M}$.*

We now define bisimulations modulo an atom p, similar to the ones from [6,24], where uniform interpolation is studied on the basis of bisimulation quantifiers. While those papers focus on purely semantic methods, we embed the semantic tools of bisimulation into our constructive proof-theoretic approach in Sect. 3. Our bisimulations behave largely like standard bisimulations except they do not have to preserve the truth of formulas with occurrences of p.

Definition 10 (Bisimilarity). *A* bisimulation up to an atom p between models $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ *is a non-empty relation $Z \subseteq W \times W'$ such that the following hold for all $w \in W$ and $w' \in W'$ with wZw':*

atoms$_p$. *$w \in V(q)$ iff $w' \in V'(q)$ for all $q \in \mathsf{Prop} \setminus \{p\}$;*
forth. *if wRv, then there exists $v' \in W'$ such that vZv' and $w'R'v'$; and*
back. *if $w'R'v'$, then there exists $v \in W$ such that vZv' and wRv.*

When wZw', we write $(\mathcal{M}, w) \sim_p (\mathcal{M}', w')$. Further, we write $(\mathcal{M}, \mathcal{I}) \sim_p (\mathcal{M}', \mathcal{I}')$ for $\mathcal{I} : X \to W$ and $\mathcal{I}' : X \to W'$ with a common domain X if there is a bisimulation Z up to p between $\mathcal{M}$ and $\mathcal{M}'$ such that $\mathcal{I}(\sigma)Z\mathcal{I}'(\sigma)$ for each $\sigma \in X$.

The main property of bisimulations is truth preservation for modal formulas. The following theorem is proved the same way as [4, Theorem 2.20].

Theorem 11. *If $(\mathcal{M}, w) \sim_p (\mathcal{M}', w')$, then for all formulas φ with $p \notin Var(\varphi)$, we have $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.*

We are interested in manipulations of treelike models that preserve bisimulation up to p, in particular, in duplicating a part of a model or replacing it with a bisimilar model.

Definition 12 (Model transformations). *Let $\mathcal{M} = (W, R, V)$ be an intransitive tree (possibly with some reflexive worlds), $\mathcal{M}_w = (W_w, R_w, V_w)$ be its subtree with root $w \in W$, and $\mathcal{N} = (W_N, R_N, V_N)$ be another tree with root $\rho_N \in W_N$. A model $\mathcal{M}' = (W', R', V')$ is the result of* replacing the subtree $\mathcal{M}_w$ with $\mathcal{N}$ in $\mathcal{M}$ *if $W' := (W \setminus W_w) \sqcup W_N$, $V'(q) := (V(q) \setminus W_w) \sqcup V_N(q)$ for all $q \in \mathsf{Prop}$, and $R' := (R \cap (W \setminus W_w)^2) \sqcup R_N \sqcup \{(v, \rho_N) \mid vRw\}$.*

A model $\mathcal{M}'' = (W'', R'', V'')$ is the result of duplicating (cloning) $\mathcal{M}_w$ *in $\mathcal{M}$ if another copy*[4] *$\mathcal{M}^c_w = (W^c_w, R^c_w, V^c_w)$ of $\mathcal{M}_w$ is inserted alongside (as a subtree of) $\mathcal{M}_w$, i.e., if $W'' := W \sqcup W^c_w$, $V''(q) := V(q) \sqcup V^c_w(q)$ for all $q \in$ Prop, and, in case of duplicating, $R'' := R \sqcup R^c_w \sqcup \{(v, w^c) \mid vRw\}$ (in case of cloning, $R'' := R \sqcup R^c_w \sqcup \{(w, w^c)\}$). Finally, for a reflexive world w, the result of* unraveling $\mathcal{M}_w$ *in $\mathcal{M}$ is obtained by first cloning $\mathcal{M}_w$ in $\mathcal{M}$ and then removing the reflexive loop (w, w) from the accessibility relation.*

Lemma 13. *In the setup from Definition 12, let $Z \subseteq W_N \times W_w$ be a bisimulation demonstrating that $(\mathcal{N}, \rho_N) \sim_p (\mathcal{M}_w, w)$. Then, for $\mathcal{M}'$ obtained by*

[4] Here $v^c := (v, c)$, $W^c_w := \{v^c \mid v \in W_w\}$, $R^c_w := \{(v^c, u^c) \mid (v, u) \in R_w\}$, and $V^c_w(q) := \{v^c \mid v \in V_w(q)\}$.

replacing $\mathcal{M}_w$ *with* $\mathcal{N}$ *in* $\mathcal{M}$ *we have that* $(\mathcal{M}', v) \sim_p (\mathcal{M}, v)$ *for all* $v \in W \setminus W_w$ *and that* $(\mathcal{M}', u_N) \sim_p (\mathcal{M}, u)$ *whenever* $u_N Z u$*. Moreover, if both* $\mathcal{M}$ *and* $\mathcal{N}$ *are* K*-models (*D*-models,* T*-models), then so is* $\mathcal{M}'$*.*

For $\mathcal{M}''$ *obtained by duplicating* $\mathcal{M}_w$ *in* $\mathcal{M}$*, we have* $(\mathcal{M}'', v) \sim_p (\mathcal{M}, v)$ *for all* $v \in W$ *and, in addition,* $(\mathcal{M}'', u^c) \sim_p (\mathcal{M}, u)$ *for all* $u \in W_w$*. If* $\mathcal{M}$ *is a* K*-model (*D*-model,* T*-model) not rooted at* w*, so is* $\mathcal{M}''$*. The same holds for cloning and unraveling if* wRw *except that unraveling does not preserve* T*-models.*

Proof. It is easy to see that $Z' := \{(v, v) \mid v \in W \setminus W_w\} \sqcup Z$ for replacing or that $Z'' := \{(v, v) \mid v \in W\} \sqcup \{(u^c, u) \mid u \in W_w\}$ for duplicating, cloning, and unraveling witnesses all the stated bisimilarities in each respective case. Both the tree structure and reflexivity of worlds are preserved by all operations other than unraveling, which turns a reflexive w into an irreflexive world, violating T-model requirements. Seriality is preserved by all operations. □

3 Uniform Interpolation for Nested Sequents

In this section we prove the UIP for K, T, and D via NK, NT, and ND. We define two new notions of UIPs for nested sequents that involve Kripke semantics: the nested-sequent UIP (NUIP) in Definition 22 that closely follows the structure of the UIP and the more convenient to use bisimulation NUIP (BNUIP) in Definition 25. Lemma 24 and Corollary 28 extract back the standard definition of the UIP.

Definition 14 (UIP). *A logic* L *in a language containing an implication* $\to$ *and Boolean constants* $\bot$ *and* $\top$ *(primary or defined) has the* uniform interpolation property*, or* UIP*, if for every formula* φ *in the logic and atom* p*, there exist formulas* $\forall p\varphi$ *and* $\exists p\varphi$ *such that*

(i) $Var(\exists p\varphi) \subseteq Var(\varphi) \setminus \{p\}$ *and* $Var(\forall p\varphi) \subseteq Var(\varphi) \setminus \{p\}$*,*
(ii) $\vdash_{\mathsf{L}} \varphi \to \exists p\varphi$ *and* $\vdash_{\mathsf{L}} \forall p\varphi \to \varphi$*, and*
(iii) for each formula ψ *with* $p \notin Var(\psi)$*:*

$$\vdash_{\mathsf{L}} \varphi \to \psi \quad \Rightarrow \quad \vdash_{\mathsf{L}} \exists p\varphi \to \psi \qquad \textit{and} \qquad \vdash_{\mathsf{L}} \psi \to \varphi \quad \Rightarrow \quad \vdash_{\mathsf{L}} \psi \to \forall p\varphi.$$

For classical-based logics, the existence of left-interpolants ensures the existence of right-interpolants, and vice versa (e.g., $\exists p\varphi := \neg\forall p\overline{\varphi}$). Thus, from now on, we focus on $\forall p\varphi$. In the following, we import some notation from [15].

Definition 15. Multiformulas *are defined by* $\mho ::= \sigma : \varphi \mid (\mho \owedge \mho) \mid (\mho \ovee \mho)$*, where* σ *is a label and* φ *is a formula. We write* $\mathcal{L}(\mho)$ *for the set of labels in* $\mho$*.*

Definition 16 (Suitability). *A multiworld interpretation* $\mathcal{I}$ *of a sequent* Γ *into a model* $\mathcal{M}$ *is* suitable for a multiformula $\mho$ *if* $\mathcal{L}(\mho) \subseteq \mathcal{L}(\Gamma)$*, in which case we call it a* multiworld interpretation of $\mho$ into $\mathcal{M}$*.*

Definition 17 (Truth for multiformulas). *Let $\mathcal{I}$ be a multiworld interpretation of a multiformula $\mho$ into a model $\mathcal{M}$. Define $\mathcal{M}, \mathcal{I} \models \mho$ recursively as:*

$$\begin{array}{lll} \mathcal{M}, \mathcal{I} \models \sigma : \varphi & iff & \mathcal{M}, \mathcal{I}(\sigma) \models \varphi, \\ \mathcal{M}, \mathcal{I} \models \mho_1 \owedge \mho_2 & iff & \mathcal{M}, \mathcal{I} \models \mho_i \text{ for both } i = 1, 2, \\ \mathcal{M}, \mathcal{I} \models \mho_1 \ovee \mho_2 & iff & \mathcal{M}, \mathcal{I} \models \mho_i \text{ for at least one } i = 1, 2. \end{array}$$

Since $\mathcal{L}(\mho_i) \subseteq \mathcal{L}(\mho)$, $\mathcal{I}$ is also a multiworld interpretation of each $\mho_i$ into $\mathcal{M}$.

We define the label-erasing function from multiformulas to formulas, as well as multiformula equivalence and some of the latter's easily provable properties.

Definition 18. *The* label-erasing function form *from multiformulas to formulas is defined as follows:* $\mathsf{form}(\sigma : \varphi) := \varphi$, $\mathsf{form}(\mho_1 \owedge \mho_2) := \mathsf{form}(\mho_1) \wedge \mathsf{form}(\mho_2)$, *and* $\mathsf{form}(\mho_1 \ovee \mho_2) := \mathsf{form}(\mho_1) \vee \mathsf{form}(\mho_2)$.

Definition 19 (Multiformula equivalence). *Multiformulas $\mho_1$ and $\mho_2$ are* equivalent, *denoted $\mho_1 \equiv \mho_2$, iff $\mathcal{L}(\mho_1) = \mathcal{L}(\mho_2)$ and $\mathcal{M}, \mathcal{I} \vDash \mho_1 \Leftrightarrow \mathcal{M}, \mathcal{I} \vDash \mho_2$ for any multiworld interpretation $\mathcal{I}$ of $\mho_1$ into a model $\mathcal{M}$.*

Lemma 20 (Equivalence property). *For any multiformula $\mho$, label σ, and formulas φ and ψ, we have $\mho \owedge \mho \equiv \mho \ovee \mho \equiv \mho$, and $\sigma : \varphi \owedge \sigma : \psi \equiv \sigma : (\varphi \wedge \psi)$, and $\sigma : \varphi \ovee \sigma : \psi \equiv \sigma : (\varphi \vee \psi)$.*

Lemma 21 (Normal forms). *For any multiformula $\mho$, there is an equivalent multiformula $\mho^d$ ($\mho^c$) in SDNF (SCNF) such that $\mho^d$ ($\mho^c$) is a $\ovee$-disjunction ($\owedge$-conjunction) of $\owedge$-conjunctions ($\ovee$-disjunctions) of labeled formulas $\sigma : \varphi$ and each disjunct (conjunct) contains exactly one occurrence of each $\sigma \in \mathcal{L}(\mho)$.*[5]

Proof. Since $\ovee$ and $\owedge$ behave classically, one can employ the standard transformation into the DNF/CNF. In order to ensure one label per disjunct/conjunct rule, multiple labels can be combined using Lemma 20, whereas missing labels can be added in the form of $\sigma : \bot$ ($\sigma : \top$). ☐

We now introduce the uniform interpolation property for nested sequents. Here, the uniform interpolants are multiformulas instead of formulas.

Definition 22 (NUIP). *Let a nested sequent calculus* NL *be sound and complete w.r.t. a logic* L. *We say that* NL *has the* nested-sequent uniform interpolation property, *or* NUIP, *if for each nested sequent Γ and atom p there exists a multiformula $A_p(\Gamma)$, called a* nested uniform interpolant, *such that*

(i) $Var\big(A_p(\Gamma)\big) \subseteq Var(\Gamma) \setminus \{p\}$ *and* $\mathcal{L}\big(A_p(\Gamma)\big) \subseteq \mathcal{L}(\Gamma)$,
(ii) for each multiworld interpretation $\mathcal{I}$ of Γ into an L*-model $\mathcal{M}$*

$$\mathcal{M}, \mathcal{I} \models A_p(\Gamma) \quad \textit{implies} \quad \mathcal{M}, \mathcal{I} \models \Gamma,$$

[5] Here 'S' in SDNF and SCNF stands for *special* to account for the additional requirement of one occurrence per label.

(iii) for each nested sequent Σ with $p \notin Var(\Sigma)$ and $\mathcal{L}(\Sigma) = \mathcal{L}(\Gamma)$ and for each multiworld interpretation $\mathcal{I}$ of Γ into an L*-model $\mathcal{M}$,*

$$\mathcal{M}, \mathcal{I} \not\models A_p(\Gamma) \text{ and } \mathcal{M}, \mathcal{I} \not\models \Sigma \quad \text{imply} \quad \mathcal{M}', \mathcal{I}' \not\models \Gamma \text{ and } \mathcal{M}', \mathcal{I}' \not\models \Sigma$$

for some multiworld interpretation $\mathcal{I}'$ of Γ into some L*-model $\mathcal{M}'$.*

NUIP(i) ensures that interpretations of Γ are suitable for $A_p(\Gamma)$.

Remark 23. Bílková's definition in [3] differs in several ways. Apart from a minor difference in NUIP(iii), our definition involves semantic notions and uses multiformula interpolants instead of formulas.

Lemma 24. *If a nested calculus* NL *has the NUIP, then its logic* L *has the UIP.*

Proof. To show the existence of $\forall p\varphi$, consider a nested uniform interpolant $A_p(\varphi)$ of the nested sequent φ, with $\mathcal{L}(\varphi) = \{1\}$. By Lemma 21, w.l.o.g. we can assume that $A_p(\varphi) = 1 : \xi$. Let $\forall p\varphi := \xi$. We establish the UIP properties based on the corresponding NUIP properties. By NUIP(i), we have that $Var(\forall p\varphi) = Var(1 : \xi) \subseteq Var(\varphi) \setminus \{p\}$ which establishes UIP(i) (cf. Definition 14).

For UIP(ii) we use a semantic argument. Assume towards a contradiction that $\nvdash_{\mathsf{L}} \xi \to \varphi$, in which case by completeness $\mathcal{M}, w \not\models \xi \to \varphi$ for some L-model $\mathcal{M} = (W, R, V)$ and $w \in W$. Consider a multiworld interpretation $\mathcal{I}$ of sequent φ into $\mathcal{M}$ such that $\mathcal{I}(1) := w$. Then $\mathcal{M}, \mathcal{I} \models 1 : \xi$ but $\mathcal{M}, \mathcal{I} \not\models \varphi$, in contradiction to NUIP(ii). Hence, $\vdash_{\mathsf{L}} \forall p\varphi \to \varphi$ as required.

Finally, for UIP(iii), let $p \notin Var(\psi)$ and suppose $\nvdash_{\mathsf{L}} \psi \to \xi$. Once again, by completeness, $\mathcal{M}, w \not\models \psi \to \xi$ for some L-model $\mathcal{M} = (W, R, V)$ and $w \in W$. Consider the nested sequent $\overline{\psi}$, with $\mathcal{L}(\overline{\psi}) = \mathcal{L}(\varphi) = \{1\}$, and a multiworld interpretation $\mathcal{I}$ of sequent φ into $\mathcal{M}$ with $\mathcal{I}(1) := w$. Then $\mathcal{M}, \mathcal{I} \not\models 1 : \xi$ and $\mathcal{M}, \mathcal{I} \not\models \overline{\psi}$. By NUIP(iii), there must exist an L-model $\mathcal{M}'$ and a multiworld interpretation $\mathcal{I}'$ of sequent φ into $\mathcal{M}'$ such that $\mathcal{M}', \mathcal{I}' \not\models \varphi$ and $\mathcal{M}', \mathcal{I}' \not\models \overline{\psi}$. In other words, $\mathcal{M}', \mathcal{I}'(1) \not\models \varphi$ and $\mathcal{M}', \mathcal{I}'(1) \models \psi$. Thus, by soundness of L, we have $\nvdash_{\mathsf{L}} \psi \to \varphi$, thus completing the proof of UIP(iii). □

We replace NUIP(iii) with a (possibly) stronger condition (iii)′ that uses bisimulations up to p to find a model $\mathcal{M}'$:

Definition 25 (BNUIP). *A nested sequent calculus* NL *has the* bisimulation nested-sequent uniform interpolation property*, or* BNUIP*, if, in addition to conditions NUIP*(i)–(ii) *from Definition 22,*

(iii)′ for each L*-model $\mathcal{M}$ and multiworld interpretation $\mathcal{I}$ of Γ into $\mathcal{M}$, if $\mathcal{M}, \mathcal{I} \not\models A_p(\Gamma)$, then there are an* L*-model $\mathcal{M}'$ and multiworld interpretation $\mathcal{I}'$ of Γ into $\mathcal{M}'$ such that $(\mathcal{M}', \mathcal{I}') \sim_p (\mathcal{M}, \mathcal{I})$ and $\mathcal{M}', \mathcal{I}' \not\models \Gamma$.*

It easily follows from Theorem 11 that, like formulas, both nested sequents and multiformulas are invariant under bisimulations:

Lemma 26. *Let Γ ($\mho$) be a sequent (multiformula) not containing p and $\mathcal{I}$ and $\mathcal{I}'$ be multiworld interpretations of Γ ($\mho$) into $\mathcal{M}$ and $\mathcal{M}'$ respectively such that $(\mathcal{M}, \mathcal{I}) \sim_p (\mathcal{M}', \mathcal{I}')$. Then $\mathcal{M}, \mathcal{I} \models \Gamma$ iff $\mathcal{M}', \mathcal{I}' \models \Gamma$ ($\mathcal{M}, \mathcal{I} \models \mho$ iff $\mathcal{M}', \mathcal{I}' \models \mho$).*

Lemma 27. *If $\Gamma, A_p(\Gamma)$ satisfy* (iii)′ *of Definition 25, then they satisfy* (iii) *of Definition 22.*

Proof. Let Σ be a nested sequent with $p \notin Var(\Sigma)$ and $\mathcal{L}(\Sigma) = \mathcal{L}(\Gamma)$. Let $\mathcal{M}, \mathcal{I} \not\models A_p(\Gamma)$ and $\mathcal{M}, \mathcal{I} \not\models \Sigma$. By condition (iii)′ we find an L-model $\mathcal{M}'$ and $\mathcal{I}'$ from Γ into $\mathcal{M}'$ such that $(\mathcal{M}', \mathcal{I}') \sim_p (\mathcal{M}, \mathcal{I})$ and $\mathcal{M}', \mathcal{I}' \not\models \Gamma$. By Lemma 26, we also conclude $\mathcal{M}', \mathcal{I}' \not\models \Sigma$. □

Corollary 28. *If a calculus NL has the BNUIP, then its logic L has the UIP.*

3.1 Uniform Interpolation For K

Now we present our method of constructing nested uniform interpolants satisfying the BNUIP for NK. Interpolants $A_p(\Gamma)$ are defined recursively on the basis of the terminating calculus from Fig. 1. If Γ is not K-saturated, $A_p(\Gamma)$ is defined recursively in Table 1 based on the form of Γ. For rows 3–5, we assume that the formula in the left column is not K-saturated in Γ, whereas in the last row we assume $\Diamond\varphi$ not to be K-saturated w.r.t. σn in Γ.[6] Each row in the table corresponds to a rule in the proof search.

Table 1. Recursive construction of $A_p(\Gamma)$ for NK for Γ that are not K-saturated.

Γ matches	$A_p(\Gamma)$ equals
$\Gamma'\{\top\}_\sigma$	$\sigma : \top$
$\Gamma'\{p, \overline{p}\}_\sigma$	$\sigma : \top$
$\Gamma'\{\varphi \vee \psi\}$	$A_p\big(\Gamma'\{\varphi \vee \psi, \varphi, \psi\}\big)$
$\Gamma'\{\varphi \wedge \psi\}$	$A_p\big(\Gamma'\{\varphi \wedge \psi, \varphi\}\big) \owedge A_p\big(\Gamma'\{\varphi \wedge \psi, \psi\}\big)$
$\Gamma'\{\Box\varphi\}_\sigma$	$\bigotimes_{i=1}^{m} \Big(\sigma : \Box\delta_i \ovee \bigovee_{\tau \neq \sigma n} \tau : \gamma_{i,\tau}\Big)$ where n is the smallest integer such that $\sigma n \notin \mathcal{L}(\Gamma)$ and the SCNF of $A_p\big(\Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\}\big)$ is $\bigotimes_{i=1}^{m} \Big(\sigma n : \delta_i \ovee \bigovee_{\tau \neq \sigma n} \tau : \gamma_{i,\tau}\Big)$,
$\Gamma'\{\Diamond\varphi, [\Delta]_{\sigma n}\}$	$A_p\big(\Gamma'\{\Diamond\varphi, [\Delta, \varphi]\}\big)$

[6] Strictly speaking, this is a non-deterministic algorithm. Since the order does not affect our results, we do not specify it. However, it is more efficient to apply rows 1–2 of Table 1 first and row 5 last.

For K-saturated Γ, we define $A_p(\Gamma)$ recursively as follows:

$$A_p(\Gamma) := \bigovee_{\substack{\sigma:\ell \in \Gamma \\ \ell \in \mathsf{Lit}\setminus\{p,\overline{p}\}}} \sigma : \ell \quad \ovee \quad \bigovee_{\substack{\tau \in \mathcal{L}(\Gamma) \\ (\exists \psi)\tau:\Diamond\psi \in \Gamma}} \tau : \Diamond A_p^{\mathsf{form}}\left(\bigvee\nolimits_{\tau:\Diamond\psi\in\Gamma} \psi\right), \qquad (1)$$

where $A_p^{\mathsf{form}}(\Gamma) := \mathsf{form}\big(A_p(\Gamma)\big)$. Since we apply form to a multiformula $\mho$ with 1 being its only label, we have $\mathcal{M}, \mathcal{I} \models \mho$ iff $\mathcal{M}, \mathcal{I}(1) \models \mathsf{form}(\mho)$. As usual, we define the empty disjunction to be false, which here means $\bigovee \varnothing := 1 : \bot$. The construction of $A_p(\Gamma)$ is well-defined (modulo a chosen order) because it terminates w.r.t. the following ordering on nested sequents. For a nested sequent Γ, let $d(\Gamma)$ be the number of its distinct diamond subformulas. Let $\ll$ be the ordering in which the rules of NK terminate (see Lemma 5). Consider the lexicographic ordering based on the pair $(d, \ll)$. For each row in Table 1, d stays the same but the recursive calls are for premise(s) lower w.r.t. ordering $\ll$. The recursive call in step (1) for K-saturated sequents, on the other hand, decreases d because the set of diamond subformulas of $\bigvee_{\tau:\Diamond\psi\in\Gamma} \psi$ is strictly smaller than that of Γ. When $d(\Gamma) = 0$ for a K-saturated Γ, the second disjunct of the recursive call (1) is empty and, thus, no new recursive calls are generated.

Before we prove the main theorem, we provide some examples.

Example 29. Consider the sequent $\Box p, \Box\overline{p}$. We use Lemmas 20 and 21 as necessary. The algorithm for $A_p(\Box p, \Box\overline{p})$ calls the calculation of $A_p(\Box p, \Box\overline{p}, [p]_{11})$, which in turn calls $A_p(\Box p, \Box\overline{p}, [p]_{11}, [\overline{p}]_{12})$. The latter sequent is K-saturated, and the algorithm returns $1 : \bot \ovee 1 : \bot$, the first disjunct corresponding to the empty disjunction of literals other than p and $\overline{p}$ and the second one representing the absent diamond formulas. Computing its SCNF we get

$$A_p(\Box p, \Box\overline{p}, [p]_{11}, [\overline{p}]_{12}) \equiv 1 : \bot \ovee 11 : \bot \ovee 12 : \bot.$$

Applying the transformation from the penultimate line of Table 1, we first get

$$A_p(\Box p, \Box\overline{p}, [p]_{11}) = 1 : \bot \ovee 11 : \bot \ovee 1 : \Box\bot \equiv 1 : \Box\bot \ovee 11 : \bot,$$

and finally $A_p(\Box p, \Box\overline{p}) = 1 : \Box\bot \ovee 1 : \Box\bot \equiv 1 : \Box\bot$. It is easy to check that $\Box\bot$ is indeed a uniform interpolant of $\Box p \vee \Box\overline{p}$.

Example 30. Consider the nested sequent $\Gamma = \overline{p}, \Diamond q \wedge \Diamond p, [q]$. In the absence of boxes, the algorithm amounts to processing the K-saturated sequents in the leaves of the proof search tree.

$$\dfrac{\overline{p}, \Diamond q \wedge \Diamond p, \Diamond q, [q]_{11} \qquad \dfrac{\overline{p}, \Diamond q \wedge \Diamond p, \Diamond p, [q,p]_{11}}{\overline{p}, \Diamond q \wedge \Diamond p, \Diamond p, [q]_{11}}}{\overline{p}, \Diamond q \wedge \Diamond p, [q]_{11}}$$

We have

$$A_p(\overline{p}, \Diamond q \wedge \Diamond p, \Diamond q, [q]_{11}) = 11 : q \ovee 1 : \Diamond A_p^{\mathsf{form}}(q),$$
$$A_p(\overline{p}, \Diamond q \wedge \Diamond p, \Diamond p, [q,p]_{11}) = 11 : q \ovee 1 : \Diamond A_p^{\mathsf{form}}(p).$$

Since formulas $A_p^{\mathsf{form}}(q)$ and $A_p^{\mathsf{form}}(p)$ can be simplified to q and $\bot$ respectively, we obtain $A_p(\Gamma) \equiv (11 : q \otimes 1 : \Diamond q) \owedge (11 : q \otimes 1 : \Diamond \bot)$, which is equivalent to $11 : q$ since $\Diamond\bot$ can never be true. Again, it is easy to see that $11 : q$ is a bisimulation nested uniform interpolant of $\overline{p}, \Diamond q \wedge \Diamond p, [q]_{11}$ with respect to p.

Theorem 31. *The nested calculus* NK *has the BNUIP.*

Proof. BNUIP(i) is easily satisfied. To prove BNUIP(ii), let Γ be a nested sequent and $\mathcal{I}$ a multiworld interpretation of Γ into a K-model $\mathcal{M} = (W, R, V)$ such that $\mathcal{M}, \mathcal{I} \models A_p(\Gamma)$ (by BNUIP(i) $\mathcal{I}$ is suitable for $A_p(\Gamma)$). We show $\mathcal{M}, \mathcal{I} \models \Gamma$ by induction on the lexicographic ordering $(d, \ll)$. Considering the construction of $A_p(\Gamma)$, we treat the cases of Table 1 first and deal with the case of K-saturated Γ last. Cases in rows 1–2 of Table 1 are trivial. Those in rows 3, 4, and 6 are similar (see [10]), so we only discuss row 5:

Let $\Gamma = \Gamma'\{\Box\varphi\}_\sigma$, and $A_p\big(\Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\}\big) \equiv \bigowedge_{i=1}^{m} \left(\sigma n : \delta_i \otimes \bigovee_{\tau \neq \sigma n} \tau : \gamma_{i,\tau}\right)$ for some $\sigma n \notin \mathcal{L}(\Gamma)$, and

$$\mathcal{M}, \mathcal{I} \models \bigowedge_{i=1}^{m} \left(\sigma : \Box\delta_i \otimes \bigovee_{\tau \neq \sigma n} \tau : \gamma_{i,\tau}\right). \tag{2}$$

For any v with $\mathcal{I}(\sigma)Rv$, define a multiworld interpretation $\mathcal{I}_v := \mathcal{I} \sqcup \{(\sigma n, v)\}$ of $\Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\}$ into $\mathcal{M}$. By (2) we have, for each i, either $\mathcal{M}, \mathcal{I}_v(\tau) \models \gamma_{i,\tau}$ for some $\tau \in \mathcal{L}(\Gamma)$ or $\mathcal{M}, \mathcal{I}_v(\sigma n) \models \delta_i$, meaning that $\mathcal{M}, \mathcal{I}_v \models A_p(\Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\})$. By the induction hypothesis, $\mathcal{M}, \mathcal{I}_v \models \Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\}$ whenever $\mathcal{I}(\sigma)Rv$. Clearly, $\mathcal{M}, \mathcal{I} \models \Gamma$ if $\mathcal{M}, \mathcal{I}(\sigma) \models \Box\varphi$. Otherwise there exists a v such that $\mathcal{I}(\sigma)Rv$ and $\mathcal{M}, v \not\models \varphi$. For this world $\mathcal{M}, \mathcal{I}_v \models \Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\}$ implies $\mathcal{M}, \mathcal{I}_v \models \Gamma'\{\Box\varphi\}_\sigma$, which yields $\mathcal{M}, \mathcal{I} \models \Gamma$ because $\mathcal{I}_v$ agrees with $\mathcal{I}$ on all labels from Γ.

Finally, for the case when Γ is K-saturated, let $\mathcal{M}, \mathcal{I} \models A_p(\Gamma)$ from (1). Clearly, $\mathcal{M}, \mathcal{I} \models \Gamma$ if we have $\mathcal{M}, \mathcal{I}(\sigma) \models \ell$ for some $\sigma : \ell \in \Gamma$. Thus, it remains to consider the case when $\mathcal{M}, \mathcal{I}(\tau) \models \Diamond A_p^{\mathsf{form}}\big(\bigvee_{\tau:\Diamond\psi\in\Gamma} \psi\big)$ for some $\tau \in \mathcal{L}(\Gamma)$. Then $\mathcal{M}, v \models A_p^{\mathsf{form}}\big(\bigvee_{\tau:\Diamond\psi\in\Gamma} \psi\big)$ for some v such that $\mathcal{I}(\tau)Rv$ and, accordingly, $\mathcal{M}, \mathcal{J} \models A_p\big(\bigvee_{\tau:\Diamond\psi\in\Gamma} \psi\big)$ for $\mathcal{J} := \{(1, v)\}$. By induction hypothesis (for a smaller d), $\mathcal{M}, \mathcal{J} \models \bigvee_{\tau:\Diamond\psi\in\Gamma} \psi$, and, hence, $\mathcal{M}, v \models \psi$ for some $\tau : \Diamond\psi \in \Gamma$. Now $\mathcal{M}, \mathcal{I} \models \Gamma$ follows from $\mathcal{I}(\tau)Rv$. This case concludes the proof for BNUIP(ii).

It only remains to prove BNUIP(iii)$'$. Let $\mathcal{I}$ be a multiworld interpretation of Γ into a K-model $\mathcal{M}$ such that $\mathcal{M}, \mathcal{I} \not\models A_p(\Gamma)$. We must find another multiworld interpretation $\mathcal{I}'$ into some K-model $\mathcal{M}'$ such that $(\mathcal{M}', \mathcal{I}') \sim_p (\mathcal{M}, \mathcal{I})$ and $\mathcal{M}', \mathcal{I}' \not\models \Gamma$. We construct $\mathcal{M}'$ and $\mathcal{I}'$ while simultaneously proving BNUIP(iii)$'$ by induction on the lexicographic order $(d, \ll)$.

Let Γ be K-saturated and $\mathcal{M}, \mathcal{I} \not\models A_p(\Gamma)$ for $A_p(\Gamma)$ from (1). The following steps are schematically depicted in Fig. 2 (see [10] for more details).

(1) First, we make the interpretation injective. It is easy to see (though tedious to describe in detail) that by a breadth-first recursion on nodes σ in Γ, one

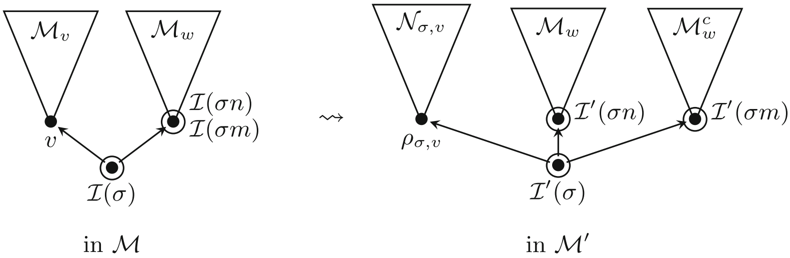


Fig. 2. Main transformations for constructing model $\mathcal{M}'$: circles are worlds in $\mathsf{Range}(\mathcal{I})$.

can duplicate $\mathcal{M}_{\mathcal{I}(\sigma n)}$ according to Definition 12 whenever $\mathcal{I}(\sigma m) = \mathcal{I}(\sigma n)$ for some $n < m$ to obtain a model $\mathcal{N}$ and an injective multiworld interpretation $\mathcal{J}$ of Γ into it such that $(\mathcal{N}, \mathcal{J}) \sim_p (\mathcal{M}, \mathcal{I})$. Thus, $\mathcal{J}(\sigma) \neq \mathcal{J}(\tau)$ whenever $\sigma \neq \tau$ and $\mathcal{N}, \mathcal{J} \not\models A_p(\Gamma)$ by Lemma 26.

(2) Then we deal with out-of-range children. A model $\mathcal{N}'$ is constructed from $\mathcal{N}$ by applying the following $\Diamond$-processing step for each node $\tau \in \mathcal{L}(\Gamma)$ that contains at least one formula of the form $\Diamond\varphi$ (nodes can be chosen in any order). Start by setting $\mathcal{N}^0 := \mathcal{N}$ and $j := 0$:
   - **$\Diamond$-processing step for τ:** Since $\mathcal{N}^j, \mathcal{J} \not\models A_p(\Gamma)$, it follows from (1) that $\mathcal{N}^j, \mathcal{J}(\tau) \not\models \Diamond A_p^{\mathsf{form}}\left(\bigvee_{\tau:\Diamond\psi\in\Gamma}\psi\right)$. Thus, $\mathcal{N}^j, v \not\models A_p^{\mathsf{form}}\left(\bigvee_{\tau:\Diamond\psi\in\Gamma}\psi\right)$ for each child v of $\mathcal{J}(\tau)$ in $\mathcal{N}^j$, and, accordingly, $\mathcal{N}_v^j, \mathcal{I}_v \not\models A_p\left(\bigvee_{\tau:\Diamond\psi\in\Gamma}\psi\right)$ for the multiworld interpretation $\mathcal{I}_v := \{(1, v)\}$ of sequent $\bigvee_{\tau:\Diamond\psi\in\Gamma}\psi$ into the subtree $\mathcal{N}_v^j$ of $\mathcal{N}^j$ with root v. By the induction hypothesis for a smaller d, there exists a K-model $\mathcal{N}_{\tau,v}$ with root $\rho_{\tau,v}$ such that $(\mathcal{N}_v^j, v) \sim_p (\mathcal{N}_{\tau,v}, \rho_{\tau,v})$ and $\mathcal{N}_{\tau,v}, \rho_{\tau,v} \not\models \bigvee_{\tau:\Diamond\psi\in\Gamma}\psi$. Let $\mathcal{N}^{j+1}$ be the result of replacing each subtree $\mathcal{N}_v^j$ for children v of $\mathcal{J}(\tau)$ not in $\mathsf{Range}(\mathcal{J})$ with $\mathcal{N}_{\tau,v}$ in $\mathcal{N}^j$ according to Definition 12. Note that all these subtrees are disjoint because the models are intransitive trees and, hence, these replacements do not interfere with one another. Note also that since $\mathsf{Range}(\mathcal{J})$ is downward closed and the roots of the replaced subtrees are outside, no world from the range is modified. Thus, $\mathcal{J}$ remains an injective interpretation into $\mathcal{N}^{j+1}$. Finally, it follows from Lemma 13 that $(\mathcal{N}^j, \mathcal{J}) \sim_p (\mathcal{N}^{j+1}, \mathcal{J})$. Hence, $\mathcal{N}^{j+1}, \mathcal{J} \not\models A_p(\Gamma)$.

   Let $\mathcal{N}' = (W', R', V')$ be the model obtained after replacements for all τ's are completed (again they do not interfere with each other). Then we have $(\mathcal{N}, \mathcal{J}) \sim_p (\mathcal{N}', \mathcal{J})$ and, for each out-of-range child v of $\mathcal{J}(\tau)$ in $\mathcal{N}$, the world $\rho_{\tau,v}$ is a child of $\mathcal{J}(\tau)$ in $\mathcal{N}'$ and $\mathcal{N}', \rho_{\tau,v} \not\models \bigvee_{\tau:\Diamond\psi\in\Gamma}\psi$. This accounts for all children of $\mathcal{J}(\tau)$ in $\mathcal{N}'$.

(3) It remains to adjust the truth values of p. We define $\mathcal{M}' := (W', R', V'_p)$ by modifying the valuation V' of $\mathcal{N}'$ as follows. We define $V'_p(q) := V'(q)$ for $q \neq p$. And for $q = p$ we define:

$$V'_p(p) := V'(p) \cap \left(W' \setminus \mathsf{Range}(\mathcal{J})\right) \sqcup \{v \in W' \mid \exists\sigma(v = \mathcal{J}(\sigma) \& \sigma\colon \overline{p} \in \Gamma)\}.$$

For $\mathcal{I}' := \mathcal{J}$, it immediately follows from the definition that

$$\mathcal{M}', \mathcal{I}'(\sigma) \not\models \overline{p} \text{ if } \sigma : \overline{p} \in \Gamma \quad \text{and} \quad \mathcal{M}', \mathcal{I}'(\sigma) \not\models p \text{ if } \sigma : p \in \Gamma. \tag{3}$$

Moreover, since subtrees $\mathcal{M}'_{\rho_{\tau,v}}$ are disjoint from $\mathsf{Range}(\mathcal{I}')$,

$$\mathcal{M}', \rho_{\tau,v} \not\models \psi \text{ whenever } \tau : \Diamond\psi \in \Gamma. \tag{4}$$

After these 3 steps, we have a model $(\mathcal{M}', \mathcal{I}') \sim_p (\mathcal{N}', \mathcal{J}) \sim_p (\mathcal{N}, \mathcal{J}) \sim_p (\mathcal{M}, \mathcal{I})$ that satisfies (3) and (4). It remains to prove that $\mathcal{M}', \mathcal{I}' \not\models \Gamma$ by showing that $\mathcal{M}', \mathcal{I}'(\sigma) \not\models \varphi$ for all $\sigma : \varphi \in \Gamma$, which is done by induction on the structure of φ. Each case, except for the $\Diamond$ case is easy (see [10]). So, let $\sigma : \Diamond\psi \in \Gamma$. To falsify $\Diamond\psi$ at $\mathcal{I}'(\sigma)$, we need to show that $\mathcal{M}', u \not\models \psi$ whenever $\mathcal{I}'(\sigma)R'u$. If $u = \mathcal{I}'(\sigma n)$ for some label $\sigma n \in \mathcal{L}(\Gamma)$, saturation ensures that $\sigma n : \psi \in \Gamma$, hence, $\mathcal{M}', u \not\models \psi$ by the induction hypothesis. The only other children of $\mathcal{I}'(\sigma)$ are $u = \rho_{\sigma,v}$, for which $\mathcal{M}', u \not\models \psi$ follows from (4). This completes the proof of BNUIP(iii)$'$ for K-saturated sequents.

To conclude the proof of BNUIP(iii)$'$, we have to treat all sequents that are not K-saturated based on Table 1. Here, the only non-trivial case is the $\Box$ case. The other cases are easy (see [10]). Assume $\mathcal{M}, \mathcal{I} \not\models A_p(\Gamma'\{\Box\varphi\}_\sigma)$, i.e.,

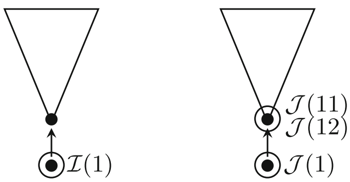
$$\mathcal{M}, \mathcal{I} \not\models \bigotimes_{i=1}^{m} \left(\sigma : \Box\delta_i \otimes \bigotimes_{\tau \neq \sigma n} \tau : \gamma_{i,\tau} \right) \tag{5}$$

where

$$A_p\big(\Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\}\big) \equiv \bigotimes_{i=1}^{m} \left(\sigma n : \delta_i \otimes \bigotimes_{\tau \neq \sigma n} \tau : \gamma_{i,\tau} \right). \tag{6}$$

By (5), for some i we have $\mathcal{M}, \mathcal{I}(\sigma) \not\models \Box\delta_i$ and $\mathcal{M}, \mathcal{I}(\tau) \not\models \gamma_{i,\tau}$ for all $\tau \neq \sigma n$. The former means that $\mathcal{M}, v \not\models \delta_i$ for some v such that $\mathcal{I}(\sigma)Rv$. Therefore, a multiworld interpretation $\mathcal{J} := \mathcal{I} \sqcup \{(\sigma n, v)\}$ of $\Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\}$ into $\mathcal{M}$ falsifies (6), and, by the induction hypothesis, there is a multiworld interpretation $\mathcal{J}'$ into a K-model $\mathcal{M}'$ such that $(\mathcal{M}', \mathcal{J}') \sim_p (\mathcal{M}, \mathcal{J})$ and $\mathcal{M}', \mathcal{J}' \not\models \Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\}$. For $\mathcal{I}' := \mathcal{J}' \restriction \mathsf{Dom}(\mathcal{I})$, we have $(\mathcal{M}, \mathcal{I}) \sim_p (\mathcal{M}', \mathcal{I}')$ and $\mathcal{M}', \mathcal{I}' \not\models \Gamma'\{\Box\varphi\}_\sigma$ because all formulas from $\Gamma'\{\Box\varphi\}_\sigma$ are present in $\Gamma'\{\Box\varphi, [\varphi]_{\sigma n}\}$. ◻

Example 32. As shown in Example 29, $A_p(\Box p, \Box\overline{p}) = 1 : \Box\bot$. To see the importance of injectivity in BNUIP(iii)$'$, suppose $\mathcal{M}, \mathcal{I} \not\models 1 : \Box\bot$, i.e., $\mathcal{I}(1)$ has at least one child. Assume this is the only child, as in a model depicted on the left:



For a saturation $\Box p, \Box\overline{p}, [p]_{11}, [\overline{p}]_{12}$ of this sequent, we found an interpolant in SCNF: namely, $1 : \bot \otimes 11 : \bot \otimes 12 : \bot$. A multiworld interpretation $\mathcal{J}$ mapping both 11 and 12 to the only child of $\mathcal{J}(1) := \mathcal{I}(1)$ yields the picture on the right. Clearly, the SCNF is false, $\mathcal{M}, \mathcal{J} \not\models 1 : \bot \otimes 11 : \bot \otimes 12 : \bot$. But, without forcing $\mathcal{J}$ to be injective, it is impossible to make $\Box p, \Box \overline{p}$ false at $\mathcal{J}(1)$: whichever truth value p has at $\mathcal{J}(11)$, it makes one of the boxes true.

3.2 Uniform Interpolation For D and T

The proof for K can be adjusted to prove the same result for D and T.

Theorem 33. *The nested sequent calculi* ND *and* NT *have the BNUIP.*

Proof. We follow the structure of the proof in Theorem 31 and only describe deviations from it. If Γ is not D-/T-saturated, then cases in Table 1 are appended with the bottom (top) row of Table 2, which is applied only if $\Diamond\varphi$ is not D-/T-saturated in Γ. For D-/T-saturated Γ, define $A_p(\Gamma)$ by (1) as before. BNUIP(i) is clearly satisfied by either row in Table 2.

Table 2. Additional recursive rules for constructing $A_p(\Gamma)$ for Γ that are not T-saturated (top row) or not D-saturated (bottom row).

Γ matches	$A_p(\Gamma)$ equals
$\Gamma'\{\Diamond\varphi\}$ in logic T	$A_p(\Gamma'\{\Diamond\varphi, \varphi\})$
$\Gamma'\{\Diamond\varphi\}_\sigma$ in logic D	$\bigovee_{i=1}^{m} \left(\sigma : \Diamond\delta_i \owedge \bigowedge_{\tau \neq \sigma 1} \tau : \gamma_{i,\tau}\right)$ where the SDNF of $A_p(\Gamma'\{\Diamond\varphi, [\varphi]_{\sigma 1}\})$ is $\bigovee_{i=1}^{m} \left(\sigma 1 : \delta_i \owedge \bigowedge_{\tau \neq \sigma 1} \tau : \gamma_{i,\tau}\right)$

Let us first show BNUIP(ii) for NT. Although T-models are reflexive, this does not affect the reasoning for either saturated sequents or non-saturated box formulas. The only new case is applying the top row of Table 2 to a non-T-saturated $\sigma : \Diamond\varphi$ in Γ. Assume $\mathcal{M}, \mathcal{I} \models A_p(\Gamma'\{\Diamond\varphi, \varphi\}_\sigma)$ for a T-model $\mathcal{M}$. By the induction hypothesis, $\mathcal{M}, \mathcal{I} \models \Gamma'\{\Diamond\varphi, \varphi\}_\sigma$. Since $\mathcal{M}, \mathcal{I}(\sigma) \models \varphi$ implies $\mathcal{M}, \mathcal{I}(\sigma) \models \Diamond\varphi$ by reflexivity, the desired $\mathcal{M}, \mathcal{I} \models \Gamma'\{\Diamond\varphi\}_\sigma$ follows.

For BNUIP(iii)$'$ for T-saturated sequents, we have to modify the construction in step (1) on p. 12 of an injective multiworld interpretation $\mathcal{J}$ into a new T-model $\mathcal{N}$ out of the given $\mathcal{I}$ into $\mathcal{M}$ where $\mathcal{M}, \mathcal{I} \not\models A_p(\Gamma)$. In the case of K, there could be only one situation of σm conflated with some already processed τ: namely, when $\tau = \sigma n$ is a sibling. This can still happen for T-models and is processed the same way. But, due to reflexivity, there is now another possibility: conflating with the parent $\tau = \sigma$. In this case, cloning is used instead of duplication, which produces a bisimilar T-model by Lemma 13. Having reflexive rather than irreflexive intransitive trees in step (2) on p. 12 does not affect the argument. The proof that $\mathcal{M}', \mathcal{I}' \not\models \Gamma$ for the given T-saturated Γ in step (3) on p. 13 requires an adjustment only for the case of $\sigma : \Diamond\psi \in \Gamma$: it is additionally necessary to show that $\mathcal{M}', \mathcal{I}'(\sigma) \not\models \psi$ for the reflexive loop at $\mathcal{I}'(\sigma)$. This is resolved by observing that $\sigma : \psi \in \Gamma$ due to T-saturation and, hence, ψ must also be false in $\mathcal{I}'(\sigma)$ by the induction hypothesis.

Finally, for BNUIP(iii)′ for non-T-saturated sequents, a new case comes from the top row of Table 2, but $\mathcal{M}', \mathcal{I}' \not\models \Gamma'\{\Diamond\varphi, \varphi\}$ obtained by the IH directly implies $\mathcal{M}', \mathcal{I}' \not\models \Gamma'\{\Diamond\varphi\}$. This completes the proof of the BNUIP for NT.

For BNUIP(ii) for ND, the only new case is applying the bottom row of Table 2 to a non-D-saturated $\sigma : \Diamond\varphi$ in $\Gamma = \Gamma'\{\Diamond\varphi\}_\sigma$. So let us assume $\mathcal{M}, \mathcal{I} \models \bigovee_{i=1}^{m} \left(\sigma : \Diamond\delta_i \owedge \bigowedge_{\tau \neq \sigma 1} \tau : \gamma_{i,\tau} \right)$ for some multiworld interpretation $\mathcal{I}$ into a D-model $\mathcal{M} = (W, R, V)$ such that the SDNF of $A_p(\Gamma'\{\Diamond\varphi, [\varphi]_{\sigma 1}\})$ equals $\bigovee_{i=1}^{m} \left(\sigma 1 : \delta_i \owedge \bigowedge_{\tau \neq \sigma 1} \tau : \gamma_{i,\tau} \right)$. Therefore, for some i we have $\mathcal{M}, \mathcal{I}(\tau) \models \gamma_{i,\tau}$ for all $\tau \in \mathcal{L}(\Gamma)$ and $\mathcal{M}, \mathcal{I}(\sigma) \models \Diamond\delta_i$. Therefore, $\mathcal{M}, v \models \delta_i$ for some v such that $\mathcal{I}(\sigma) R v$. Formula $\Diamond\varphi$ is not D-saturated in $\Gamma'\{\Diamond\varphi\}_\sigma$, so $\mathcal{I}_v := \mathcal{I} \sqcup \{(\sigma 1, v)\}$ is a multiworld interpretation of $\Gamma'\{\Diamond\varphi, [\varphi]_{\sigma 1}\}$ into $\mathcal{M}$. Moreover, we have $\mathcal{M}, \mathcal{I}_v \models A_p(\Gamma'\{\Diamond\varphi, [\varphi]_{\sigma 1}\})$. By induction hypothesis, $\mathcal{M}, \mathcal{I}_v \models \Gamma'\{\Diamond\varphi, [\varphi]_{\sigma 1}\}$, from which it easily follows that $\mathcal{M}, \mathcal{I} \models \Gamma'\{\Diamond\varphi\}_\sigma$.

For BNUIP(iii)′ for a D-saturated sequent Γ, we change step (1) in such a way that not only is the multiworld interpretation $\mathcal{J}$ injective, but $\mathsf{Range}(\mathcal{J})$ contains only irreflexive worlds. Injectivity is obtained in a similar way as done for T using duplication and cloning to obtain a bisimilar D-model $\mathcal{M}''$ by Lemma 13 with injective multiworld interpretation, say $\mathcal{J}$. So $(\mathcal{M}'', \mathcal{J}) \sim_p (\mathcal{M}, \mathcal{I})$, with injective $\mathcal{J}$. To ensure that the multiworld interpretation only maps to irreflexive worlds, we repeatedly unravel subtrees rooted in reflexive worlds from $\mathsf{Range}(\mathcal{J})$ while keeping the same multiworld interpretation $\mathcal{J}$. Since each unraveling decreases the number of reflexive worlds in $\mathsf{Range}(\mathcal{J})$, this process terminates yielding a model $\mathcal{M}'$ that is a D-model and satisfies $(\mathcal{M}', \mathcal{J}) \sim_p (\mathcal{M}, \mathcal{I})$ by Lemma 13. The replacements of step (2) preserve D-models by Lemma 13 and step (3) requires no changes either. Note that in steps (2) and (3) we do not change the range of $\mathcal{I}' := \mathcal{J}$, so it still only maps to irreflexive worlds. We need this construction in the proof that $\mathcal{M}', \mathcal{I}' \not\models \Gamma$ for case $\sigma : \Diamond\psi \in \Gamma$, where the argument for $\mathcal{M}', \mathcal{I}'(\sigma) \not\models \Diamond\psi$ now works the same way as in K since $\mathcal{I}'(\sigma)$ is irreflexive by construction.

The only remaining new case is the application of the bottom row of Table 2 for a non-D-saturated $\sigma : \Diamond\varphi$, i.e., when node σ is a leaf of the sequent tree, in BNUIP(iii)′. Let $\mathcal{M}, \mathcal{I} \not\models \bigovee_{i=1}^{m} \left(\sigma : \Diamond\delta_i \owedge \bigowedge_{\tau \neq \sigma 1} \tau : \gamma_{i,\tau} \right)$. By seriality of $\mathcal{M}$, there is a world $v \in W$ such that $\mathcal{I}(\sigma) R v$. Then $\mathcal{J} := \mathcal{I}' \sqcup \{(\sigma 1, v)\}$ is a multiworld interpretation of $\Gamma'\{\Diamond\varphi, [\varphi]_{\sigma 1}\}$ into $\mathcal{M}$. Moreover, we have $\mathcal{M}, \mathcal{J} \not\models \bigovee_{i=1}^{m} \left(\sigma 1 : \delta_i \owedge \bigowedge_{\tau \neq \sigma 1} \tau : \gamma_{i,\tau} \right)$. By induction hypothesis, there is a multiworld interpretation $\mathcal{J}'$ of $\Gamma'\{\Diamond\varphi, [\varphi]_{\sigma 1}\}$ into some D-model $\mathcal{M}'$ such that $(\mathcal{M}', \mathcal{J}') \sim_p (\mathcal{M}, \mathcal{J})$ and $\mathcal{M}', \mathcal{J}' \not\models \Gamma'\{\Diamond\varphi, [\varphi]_{\sigma 1}\}$. Similar to the case of $\Box\varphi$ for K, restricting this $\mathcal{J}'$ to the labels of Γ yields a multiworld interpretation bisimilar to $\mathcal{I}$ and refuting $\Gamma = \Gamma'\{\Diamond\varphi\}_\sigma$. $\square$

Corollary 34. *Logics* K*,* D*, and* T *have the uniform interpolation property.*

4 Conclusion

We developed a constructive method of proving uniform interpolation based on nested sequent calculi. While this is an important and natural step to further utilize these formalisms, much remains to be done. This method works well for the non-transitive logics K, D, and T but meets with difficulties, e.g., for S5, which is also known to enjoy uniform interpolation. In [10], we successfully adapted the method to hypersequents to cover S5. There are other logics in the so-called modal cube between K and S5 with the UIP, for which it remains to find the right formalism and adaptation of our method. Another natural direction of future work is intermediate logics, where exactly seven logics are known to have the UIP.

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