Teaching the derivative using arrow graphs in GeoGebra

Rogier Bos and Laura Brinks

Utrecht University, Freudenthal Institute, Utrecht, the Netherlands; r.d.bos@uu.nl

Teaching the derivative is usually supported by the visual context of graphs and tangent lines. However, this geometric meaning-making is somewhat indirect, hard to grasp for students, and easily forgotten. We report on the first cycle of a design-based study in which we introduce arrow graphs as an additional geometric context to provide meaning to the instantaneous rate of change as an enlargement factor with respect to a local focus. We outline a learning trajectory deploying interactive dynamic visualizations designed in GeoGebra. Our results show the challenges of this approach and suggest several ways in which the design can be improved for the next design cycle.

Keywords: calculus education, arrow graphs, dynamic geometry environment.

The derivative is usually introduced geometrically as the slope of the tangent line to the curve. Though this supports a meaningful image of the concept, we believe it is not the optimal picture for understanding the derivative as an instantaneous rate of change. Consider a closed vat with an ideal gas with a valve that controls the pressure. The volume can be described as a function of the pressure $V(p) = \frac{c}{p}$. The derivative $V'(p) = -\frac{c}{p^2}$ expresses the *sensitivity* of volume as a function of the pressure, e.g., when pressure is low, the impact on the volume of an increase is large and negative. This is observable as the slope of the graph of V is steep and negative for small values of p. However, the sense of rate is not so easily associated with steepness. We hypothesize that the sense of rate may come more naturally in the context of enlargement.

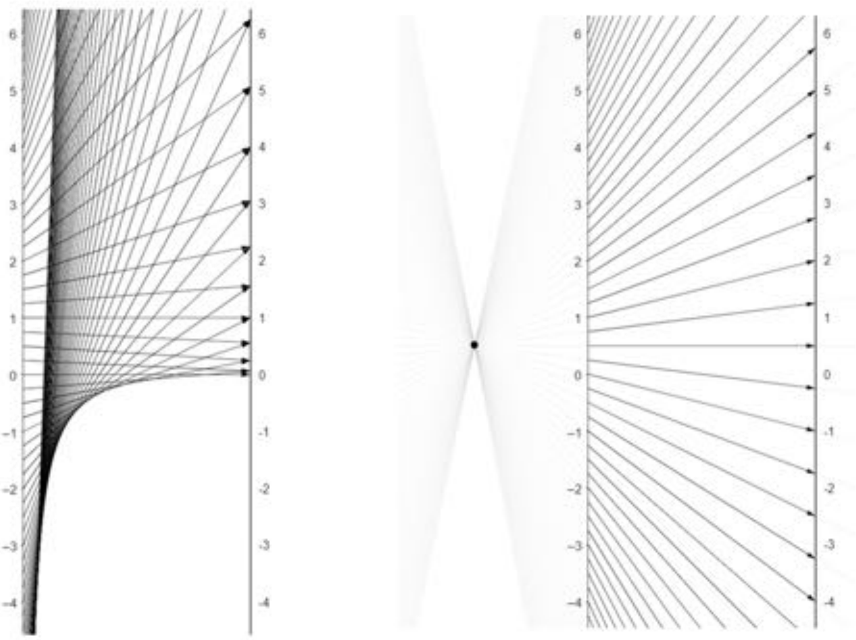

Figure 1. Left: an arrow graph for $f(x) = x^2$. Right: an arrow graph for $g(x) = 3x - 1$ with a focus

In recent work, Wei and collaborators studied how to promote functional thinking using arrow graphs—also known as the parallel axes representations or nomograms (Wei et al., 2024)—which were previously studied by Nachmias and Arcavi (1990). In an arrow graph, a function is represented as a family of arrows from input values to corresponding output values (see Figure 1). Arcavi pointed out that linear functions correspond to arrow graphs where the arrows intersect at one point—when needed the arrows are extended to a line. This point is called the focus. When the rate of change equals 1, the focus is “at infinity” and the arrows are parallel.

As a consequence, the rate of change for a linear function can be interpreted geometrically as an enlargement factor in the arrow graph. An interval on the input axis is enlarged to an interval on the output axis with respect to the focus (see Figure 2, left). The rate of change corresponds precisely to the enlargement factor. The main aim of our study is to investigate whether teaching this new geometric interpretation of rate of change, in addition to the usual one as slope, supports students’ sense-making.

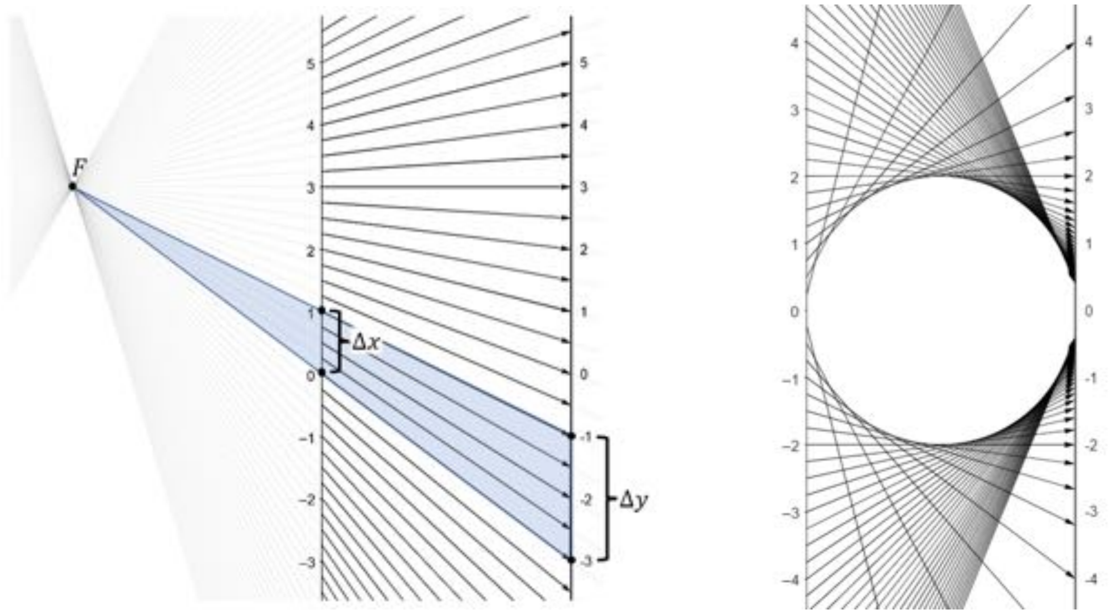

Figure 2. Left: The rate of change of $f(x) = 2x - 3$ equals the enlargement factor $\frac{\Delta y}{\Delta x} = 2$ of the interval $[0, 1]$ to $[-3, -1]$ with respect to the focus F in the arrow graph. Right: the circle as an emerging enveloping curve for the function $g(x) = \frac{4}{x}$ in the arrow graph.

More generally, the derivative of a function can be interpreted in its arrow graph. Whenever one considers the arrow graph of non-linear functions certain enveloping curves emerge in the picture (see Figure 2, right). These curves turn out to be related to the derivative in a surprising way, that we explain in a moment. A function is differentiable only if it is locally linear. In the graph, this means that around a point $(a, f(a))$, it can be approximated (in a certain technical sense) by a line, called the tangent line. Alternatively, in the arrow graph, this means that on a small interval around a on the input axis, the arrows intersect approximately at one point. The smaller the interval the sharper the point emerges, and the associated limit point we call the local focus (see Figure 3). The value of the derivative $f'(a)$ can be interpreted as the enlargement factor associated with this local focus. By construction, the local focuses form the mentioned enveloping curve. Our aim extends to studying how to teach the instantaneous rate of change and derivative using the arrow graph in this way.

Drawing arrow graphs by hand is a time-consuming endeavor. In GeoGebra, we find suitable software to outsource this to. Moreover, the dynamic features of GeoGebra allow us to vary the function, the number of arrows, and covary information across multiple representations: the graph, the arrow graph, and the equation describing the function. In this design study, we are interested in how the dynamic features of GeoGebra in our lesson design can contribute to the students' learning.

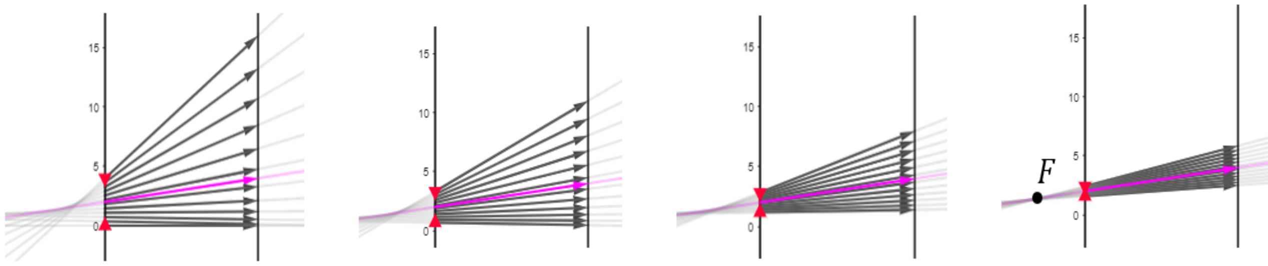

Figure 3. Arrows on a small enough interval approximately intersect in a local focus

This paper reports primarily on our original teaching designs, and also on the findings of a small pilot study, of the first cycle of a design-based research. We briefly review some literature on learning about derivatives, and on connecting multiple representations (including the arrow graph) using digital technology. Then we introduce the intervention consisting of two modules of approximately one hour. Finally, we present and reflect on the result of an implementation in a 10th-degree pre-university class.

Theoretical background

Students are usually supported in making sense of the instantaneous rate of change $\lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$ for a function f at $x = a$ by providing a geometric interpretation as the slope of the tangent line to the graph of the function at the point $(a, f(a))$. However, many students have difficulty connecting such geometric and symbolic notions, in particular grasping the limit procedure involved (Orton, 1983). Indeed, using an open, inquiry-based task, Bos et al. (2019) showed that students rather think about tangent lines and the derivative in ways not involving the limit of secant lines. As explained in the introduction, this paper develops another geometric approach, where the limit is geometrically interpreted as a local focus becoming more sharply defined as the interval shrinks.

Teaching sequences in dynamic geometry environments for the (instantaneous) rate of change and the tangent line were proposed, amongst others, by Biza et al. (2007) in CalGeo, and by Hohenwater et al. (2008) in GeoGebra. On the one hand, these sequences involve the task of zooming in on a graph to observe how a differentiable function is locally linear. This supports the fundamental insight that the graph of such a function can be approximated by the tangent line and hence the function by a linear function. On the other hand, given a point A on a graph, students are invited to move a second point B on the curve towards point A, such that the secant line AB approximates the tangent line at A. By showing the covarying values of the coordinates, Δx, Δy, and the difference quotient, students gain insight in the limit process that constitutes the differential quotient.

We propose a new approach to teaching these concepts in a dynamic geometry environment, introducing an additional representation: the arrow graph. This approach was inspired on Zandieh's theoretical framework for the concept of the derivative, as is explained in the second author's master thesis (Brinks, 2024). We pose the following research question: how can interactive tasks in GeoGebra, with a central role for the arrow graphs, promote the meaning-making of the derivative?

Method

The intervention consisted of two modules: the first focusing on arrow graphs of linear functions, and the second on the interpretation of the derivative in arrow graphs; moving from the linear situation with a focus to the locally linear situation with a local focus. The first module began by revising linear functions and their graphs, which were considered pre-knowledge. Next, arrow graphs were introduced.

Let us highlight two tasks aimed at connecting three representations of linear functions: the arrow graph, the graph, and an equation. In task 1.4a (see Figure 4), students vary the parameters of a linear function by dragging the focus (the red point on the left). Simultaneously, students can observe how the graph (the green line on the right) and the equation (top right) respond to their hand movement. In the arrow graph in pink an interval of size 1 is indicated on the input axis, together with its image and its size on the output axis. This is shown to invite students to connect the size of the enlarged interval to the rate of change in the equation, which in turn expresses the slope of the line, hence providing sense to the rate of change in the arrow graph representation. Later tasks allow students to discover how vertical movement of the focus leaves this number unchanged, whereas horizontal movement does impact it.

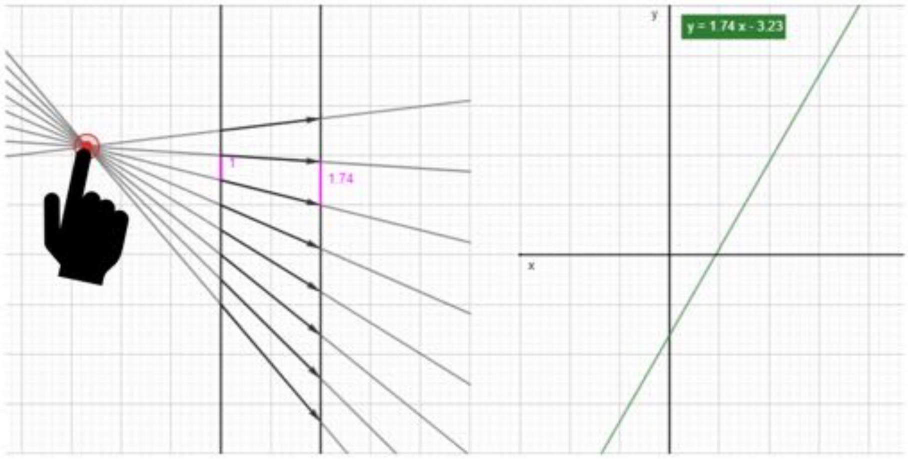

Figure 4. Task 1.4a: Connect the enlargement factor associated with the focus of the arrow graph to the slope of the graph.

In task 1.4b students again move the focus (see Figure 5), but now the distance to the input axis is indicated by an orange dashed line and the distance to the output axis by a blue line. Whereas in the previous task, the connection between the rate of change and the shown information in the arrow graph was direct, in this task students need to realize that dividing the distance to the output axis by the distance to the input axis is another way to compute the enlargement factor, that is, the rate of change. In later tasks, students explore in more depth how the sign and size of the rate of change are determined by the horizontal position of the focus. For example, when the focus is between the axes the rate of change is negative and otherwise positive. The way linear behavior presents itself in three ways in three different representations is summarized in Table 1, which was part of a theoretical section of the learning materials.

 ISBN 978-88-6629-080-3

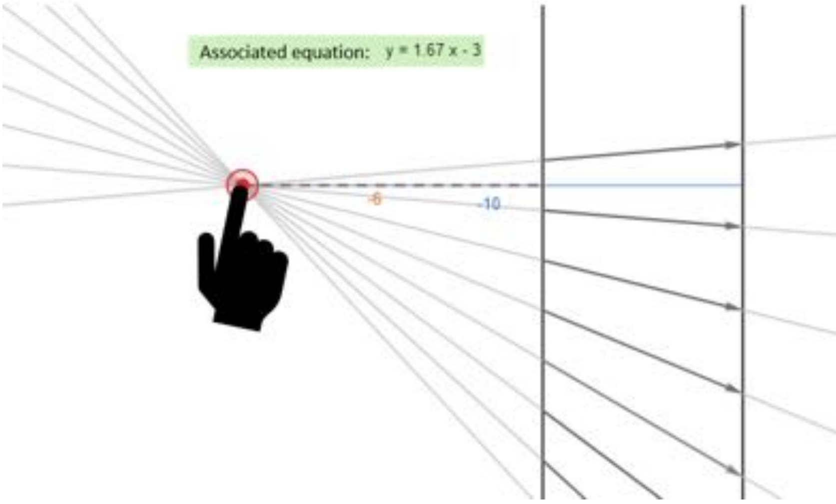

Figure 5. Task 1.4b: Connect the distances of the focus to the vertical axes in the arrow graph to the rate of change in the equation

Table 1. The relation of linear behavior in three representations

Equation: $y = a\,x + b$	**Graph**: line	**Arrow graph**: has focus
a is the rate of change	a equals the slope of the line	a is the enlargement factor
b is the initial value	b equals the y-coordinate of the intersection with the y-axis	b is the output for input 0.

In the second module attention shifts to non-linear functions and local focuses. We highlight three tasks. A central insight is developed in task 2.5 (see Figure 6). Students drag a slider to make the interval where the arrows are drawn smaller and smaller. As the interval shrinks, the local focus becomes more sharply delineated, as in Figure 3. This way students experience how local linearity is presented in an arrow graph. This is addressed in detail in a later theoretical section that also contains Table 2. Generally, the hypothesis is that connecting the three representations, offering three perspectives and their relation, supports students to make sense of the underlying concepts.

Table 2. The relation of local linear behavior in three representations

The **function** is differentiable in $x = a$	The **graph** is locally linear at $(a, f(a))$	The **arrow graph** has a local focus where the line through $a \rightarrow f(a)$ intersects the enveloping curve
Instantaneous rate of change $\lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$	Slope of the tangent line at $(a, f(a))$	The enlargement factor, e.g., $\frac{\text{signed distance local focus to input axis}}{\text{signed distance local focus to output axis}}$

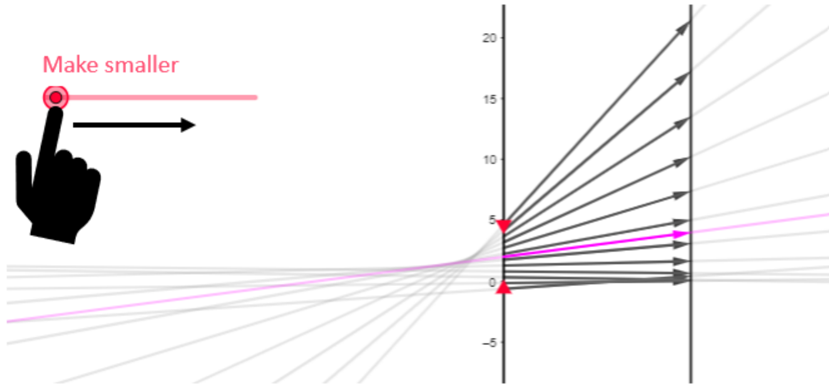

Figure 6. Task 2.5: Sliding makes the interval smaller so the local focus comes into focus

Some of the tasks in module 2 have the context of a falling object, for example, task 2.6 (see Figure 7). In task 2.6, students are invited to estimate the speed of the object after 2 seconds of falling. Students can drag two points on the input axis of the arrow graph on the left. The corresponding points and intervals in the graph on the right covary, as do the numbers in the arrow graph and in the division on the left. The goal is again to create a more intimate link between the two representations, combining the data needed for this computation from both graphs. In both Task 2.5 and Task 2.6, students manually perform part of the limit procedure, which helps them ground this essential step in the genesis of the derivative concept in a sensorimotor experience.

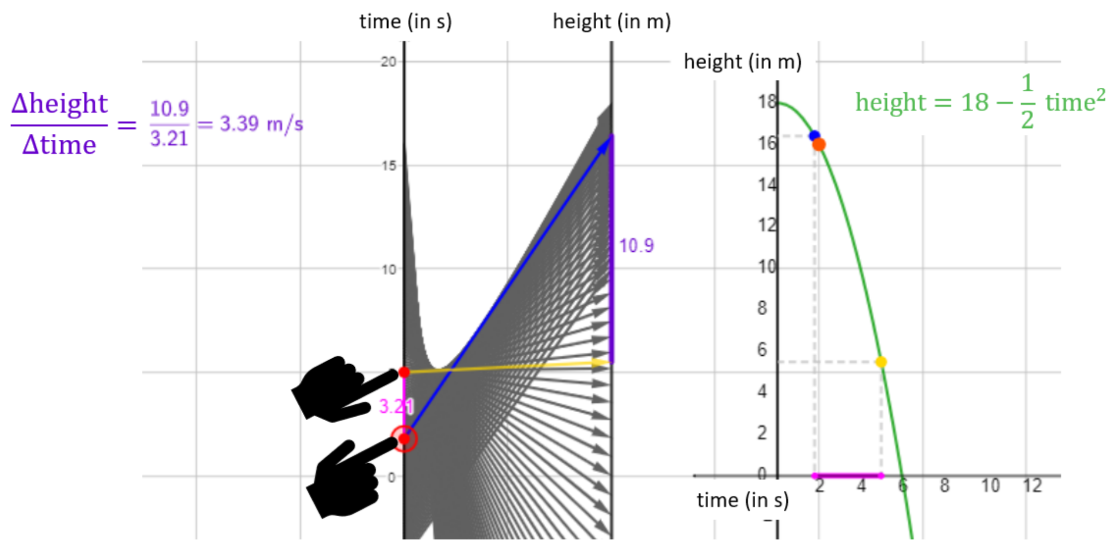

Figure 7. Task 2.6: students drag the edge points of the interval to find a good approximation of the speed after 2 seconds, with covarying information in the division on the left and graph on the right

Task 2.9 invited students to drag a point on the input value (see Figure 8). Simultaneously, the arrows (and lines through these arrows) through this point and through a point a little higher are drawn. The intersection point of these lines is an approximation of the local focus. GeoGebra is set to leave a trace of this intersection point. Hence, students can see an approximation of the enveloping curve light up in green.

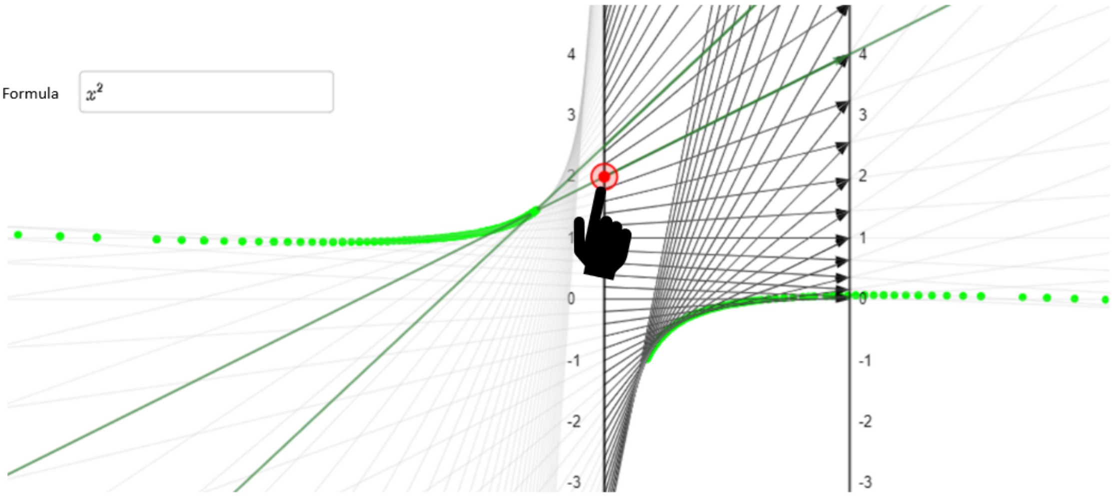

Figure 8. Task 2.9: dragging the input point gives a trace of approximate local foci

The two modules were piloted in a grade 10 class with 25 students in a social science pre-university stream. Students were allowed to collaborate. The data for this study are the answers students wrote in the answer boxes in GeoGebra. Answers were grouped for similarity and analyzed to see how performing the tasks contributed to making sense of the concepts of (instantaneous) rate of change.

Results and conclusions

We present two main conclusions, with supporting results. Firstly, we conclude that, for social science stream students, learning to work with arrow graphs and their relation to graphs and functions is challenging. This is evidenced by responses to Task 1.4b, where only 7 out of 25 students realized the slope is obtained by dividing the presented distances to the axes. The other responses to the task range from correct but superficial observations to having no clue. We found that ideas from graphs can be persistently incorrectly transferred. At the beginning of module 2, when asked why an arrow graph represented a linear function, only three students mentioned the presence of a focus, whereas many more said that “the lines [of the arrow graph] are straight” or “the arrows are straight lines”. Also, at task 2.6, even though many students managed to compute average rates of change, the transition to the instantaneous rate was mostly not understood, witnessed by only four correct computations. In task 2.9, no one managed to compute the instantaneous rate of change using the local focus. Only three students realized that the traced point was approximately the local focus. We see that these concepts and associated procedures are problematic, even when visually supported by arrow graphs. However, the types of tasks and the time spent per subject could also be better adapted to the students in the social science stream, who are generally less proficient at mathematics than those in the natural science stream.

Secondly, we conclude that the tasks should be redesigned, in particular, to make conceptual learning more visible. This would benefit conceptual discussions between students and between students and teachers, but also facilitate our analysis as researchers. In task 1.4a all students correctly observe that the pink number indicating the size of the interval on the output axis equals the slope in the graph. However, because of the task design, students have no opportunity to show that they understand why this is the case. A similar case appears in task 2.7, where some students have correct computations, but the task setting offers no means to establish whether these are accompanied by insight. Finally, some tasks should be redesigned to support improved insight. For example, in task 2.5 only two students observe that the local focus appears, whereas others just notice that the arrows are moving closer. This can easily be resolved by having separate sliders for the number of arrows and the size of the interval.

Even though the results here are slightly discouraging, we are not ready to give up on arrow graphs' role in teaching the derivative, also fuelled by their success in fostering functional thinking (Wei et al., 2024). Some factors could explain the limited success of the approach so far. Firstly, unlike working in graphs, for which students are prepared from an early age onwards, arrow graphs are new to students and the learning curve in two hours is steep. Secondly, since students in the *natural* science track have more affinity with geometry, the approach may be more fruitful for them. Therefore, the next step in this design-based research will be testing an expanded version of the series, implementing the suggestions above, on students in a natural science track. Suggestions for redesign can be found in the second author's master thesis (Brinks, 2024).

References

Biza, I., Diakoumopoulos, D. & Souyoul, A. (2007). Teaching Analysis in Dynamic Geometry Environments. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the 5th conference of European research in mathematics education* (pp. 1359–1368). http://ermeweb.free.fr/CERME5b/WG9.pdf

Bos, R. D., Doorman, L. M., & Piroi, M. (2020). Emergent models in a reinvention activity for learning the slope of a curve. *Journal of Mathematical Behavior, 59*, Article 100773. https://doi.org/10.1016/j.jmathb.2020.100773

Brinks, L. (2024). The derivative through nomograms. Master Thesis at Utrecht University. Manuscript expected summer 2024.

Hohenwarter, M., Hohenwarter, J., Kreis, Y., & Lavicza, Z. (2008). Teaching and learning calculus with free dynamic mathematics software GeoGebra. *TSG16: research and Development in the Teaching and Learning of Calculus, ICME 11*. https://orbilu.uni.lu/bitstream/10993/47219/1/ICME11-TSG16.pdf

Nachmias, R., & Arcavi, A. (1990). A Parallel Representation of Linear Functions Using a Microcomputer-Based Environment. *Journal of Computers in Mathematics and Science Teaching, 9*(4), 79–88.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics 14*, 235–250. https://doi-org/10.1007/BF00410540

Wei, H., Bos, R. & Drijvers, P. (2024). Developing Functional Thinking: From concrete to abstract through an embodied design. Submitted for publication.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education, 8*, 103–122.